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Characterization of Heat Kernel Lower Bounds for Symmetric Jump Processes

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Abstract: This paper is devoted to symmetric mixture stable-like processes on metric measure space.Assume that the process enjoys upper bounds of heat kernel,characterization of near diagonal lower bounds of heat kernel is given.Different from previous work which is usually based on regularity of elliptic harmonic functions,here we make full use of smooth properties of functions space associated with the generator of the process.

Keywords: Symmetric jump process;Heat kernel;Lower bound estimate

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1. Introduction

Let (M, d, μ) be a measurable metric space,where the support of μ is the whole space M.Consider the following symmetric regular Dirichlet form on $L^2(M; \mu)$

$$\mathcal{E}(f, g) = \int_{M\times M\setminus \mathrm{diag}} \big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\, J(dx, dy), \quad f, g \in \mathcal{F},$$

$$\mathcal{F} = \{f \in L^2(M; \mu): \mathcal{E}(f, f) < \infty\}$$

Where diag represents the diagonal set $\{(x, x): x \in M\}$.Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$be the L^2generator of Dirichlet form $(\mathcal{E}, \mathcal{F})$,and let $X \coloneqq (X_t)_{t\geq 0}$be the symmetric Hunt process generated by the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.We denote the Markov semigroup corresponding to $\mathcal{L}$by $(P_t)_{t\geq 0}$,so that there exists an exception set $\mathcal{N} \subset M$such that there is a bounded measurable function over any M.

$$P_t f(x) = \mathbb{E}^x f(X_t), \quad x \in M_0 \coloneqq M \setminus \mathcal{N}.$$

Call the non-negative measurable function $p(t, x, y)$on $M_0 \times M_0$the heat kernel corresponding to the semigroup $\{P_t\}_{t\geq 0}$,if for any $t > 0$,there is

$$\mathbb{E}^x f(X_t) = P_t f(x) = \int p(t, x, y) f(y)\, \mu(dy), \quad x \in M_0, f \in L^\infty(M; \mu).$$

In recent years,a large number of literature have given heat kernel estimates of non-local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.For example,[1]first establishes a bilateral estimate of heat kernel of symmetrically α-stable processes on d-sets,and then[2]obtains the correlation properties of heat kernel of symmetrically mixed α-stable processes in the metric measure space.[1,2]requires that $\alpha \in (0,2)$or the Scaling function it involves belongs to(0,2).Recently,[3]has established an equivalent characterization of heat kernel bilateral estimators for symmetric mixed α-stable processes in the general metric measure space,which particularly provided stability conclusions for the heat kernel to hold.The relevant conclusions on non-local Dirichlet form heat kernel can be seen in[4,5,6].

In fact,an equivalent characterization of the symmetric mixed α-stable heat kernel upper bound is also given in[3].The lower bound estimation is divided into two steps.Firstly,the estimation near the diagonal is considered.In addition to using the upper bound estimation,it usually needs to use the continuity of the elliptic harmonic function.Secondly,the Lévy system is used to provide the lower bound estimation outside the diagonal.It should be noted here that the continuity of non-local Dirichlet harmonic functions is a very profound subject,see[7,8].The purpose of this paper is to show the equivalent characterization of the lower bound estimation near the heat kernel diagonal,which is closely related to the continuity of the operator function space.

2. Lower Bound Estimation Near the Heat Kernel Diagonal

We say that the measurable metric space (M, d, μ)satisfies the volume multiplication condition (VD),if there is a constant $C_\mu \geq 1$ such that for any $x \in M$and $r > 0$,

$$V(x,2r) \leq C_\mu V(x,r). \tag{1}$$

It is easy to see that the VD condition(1)is equivalent to the existence of constants $d_2 > 0$ and $\widetilde{C_\mu} > 0$ such that for any $x \in M$ and $s \geq r > 0$,there is

$$V(x,s) \leq \widetilde{C_\mu}\left(\frac{s}{r}\right)^{d_2} V(x,r). \tag{2}$$

A measurable metric space (M, d, μ)satisfies the reversible volume multiplication condition (RVD),if there are constants $d_1 > 0$ and $c_\mu > 0$ such that for any $x \in M$and $0 < r \leq R$,

$$\frac{V(x,R)}{V(x,r)} \geq c_\mu \left(\frac{R}{r}\right)^{d_1}. \tag{3}$$

Theorem 2.1.Suppose the measurable metric space (M, d, μ)satisfies the (VD)condition,and the heat kernel upper bound (UHK)holds,that is,there are constants $\alpha > 0$, $C_0 > 0$ such that for any $t > 0$,almost everywhere $x, y \in M$,

$$p(t,x,y) \leq C_0\left(\frac{1}{V(x,t^{1/\alpha})} \wedge \frac{t}{V(x,d(x,y))d^\alpha(x,y)}\right), \tag{4}$$

At the same time,there are $p > 1$, $\beta > d/p$, $C_1 > 0$ such that for any $f \in \mathcal{F}$, $x, y \in M$holds

$$|f(x) - f(y)| \leq \frac{C_1 d^\beta(x,y)}{(V(x,d(x,y)))^{1/p}} ||\mathcal{L}^{\beta/\alpha} f||_p, \tag{5}$$

So,there exist $c_0, C_2 > 0$,such that when $d(x,y) \leq c_0 t^{1/\alpha}$,there is

$$p(t,x,y) \geq \frac{C_2}{V(x,t^{1/\alpha})}. \tag{6}$$

We first use the VD condition and(4)to prove the lower bound estimate on the diagonal,that is,for any $t > 0$,almost everywhere $x \in M$,there is

$$p(t,x,x) \geq \frac{c_0}{V(x,t^{1/\alpha})}. \tag{7}$$

In fact,according to(4),for $C > 0$,

$$\int_{M\setminus B(x,Ct^{1/\alpha})} p(t,x,y)\ \mu(dy) \leq c_1 \int_{M\setminus B(x,Ct^{1/\alpha})} \frac{t}{V(x,d(x,y))d^\alpha(x,y)}\ \mu(dy)$$

$$= c_1 \sum_{k=0}^{\infty} \int_{B(x,C2^{k+1}t^{1/\alpha})\setminus B(x,C2^k t^{1/\alpha})} \frac{t}{V(x,d(x,y))d^\alpha(x,y)}\ \mu(dy)$$

$$\leq c_1 \frac{t}{V(x,C2^k t^{1/\alpha})(C2^k t^{1/\alpha})^\alpha} \sum_{k=0}^{\infty} \int_{B(x,C2^{k+1}t^{1/\alpha})\setminus B(x,C2^k t^{1/\alpha})} \mu(dy)$$

$$\leq c_2 \sum_{k=0}^{\infty} \frac{tV(x,C2^{k+1}t^{1/\alpha})}{V(x,C2^k t^{1/\alpha})(C2^k t^{1/\alpha})^\alpha}$$

$$\leq \frac{c_3}{C^\alpha} \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} \tag{8}$$

Where $c_3 > 0$ does not depend on C.Thus,for Clarge enough,

$$\int_{M\setminus B(x,Ct^{1/\alpha})} p(t,x,y)\ \mu(dy) \leq \frac{1}{2},$$

So,for any $t > 0$,almost everywhere $x \in M$,there is

$$\int_{B(x,Ct^{1/\alpha})} p(t,x,y)\ \mu(dy) \geq \frac{1}{2}. \tag{9}$$

Next,we will explain that(7)holds true.By using Markov property and the symmetry of $p(t,x,y)$,it can be seen that,

$$p(2t,x,x) = \int_M p(t,x,y)p(t,y,x)\ \mu(dy) = \int_M p^2(t,x,y)\ \mu(dy)$$

$$\geq \int_{B(x,Ct^{1/\alpha})} p^2(t,x,y)\ \mu(dy) \geq \frac{1}{V(x,Ct^{1/\alpha})} \left(\int_{B(x,Ct^{1/\alpha})} p(t,x,y)\ \mu(dy) \right)^2$$

$$\geq \frac{1}{4V(x,Ct^{1/\alpha})},$$

Among them,the second inequality is based on the Cauchy-Schwarz inequality,and the last inequality uses(9),so(7)is proved.

Next,we use(5)to prove a lower bound estimate for non-diagonal lines.First fix $z \in M$and apply condition(9)to $f(\cdot) = p(t,\cdot,z)$,i.e.

$$|p(t,x,z) - p(t,y,z)| \leq \frac{C_4 d^\beta(x,y)}{(V(x,d(x,y)))^{1/p}} ||\mathcal{L}^{\beta/\alpha} p(t,\cdot,z)||_p \tag{10}$$

It should be pointed out that $p(t,\cdot,z) \in \mathcal{D}(\mathcal{L}^{\beta/\alpha})$by analytical properties of $e^{-t\mathcal{L}}$.

Note that

$$\mathcal{L}^{\beta/\alpha} p(t,\cdot,z) = \mathcal{L}^{\beta/\alpha} e^{-(t/2)\mathcal{L}} p(t/2,\cdot,z),$$

and

$$||\mathcal{L}^{\beta/\alpha} e^{-(t/2)\mathcal{L}}||_{p\to p} \leq c_5 t^{-\beta/\alpha},$$

therefore

$$||\mathcal{L}^{\beta/\alpha}p(t,\cdot,z)||_p \le c_5 t^{-\beta/\alpha}||p(t/2,\cdot,z)||.$$

By the Hölder inequality,we get

$$||p(t,\cdot,z)||_p \le ||p(t,\cdot,z)||_1^{1/p}||p(t,\cdot,z)||_\infty^{1-(1/p)},$$

It is further known from(4)

$$||p(t,\cdot,z)||_\infty \le \frac{c_6}{V(z,t^{1/\alpha})},$$

And since $e^{-t\mathcal{L}}$is the symmetry of Markov semigroups,we know

$$||p(t,\cdot,z)||_1 = 1,$$

Hence there is

$$||p(t,\cdot,z)||_p \le \frac{c_6}{\left(V(z,t^{1/\alpha})\right)^{1-(1/p)}},$$

Consequently

$$||\mathcal{L}^{\beta/\alpha}p(t,\cdot,z)||_p \le \frac{c_7 t^{-\beta/\alpha}}{\left(V(z,t^{1/\alpha})\right)^{1-(1/p)}}.$$

Based on(10),it can be concluded that

$$|p(t,x,z)-p(t,y,z)| \le c_8\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\beta}\left(\frac{V(z,t^{1/\alpha})}{V(x,d(x,y))}\right)^{1/p}\frac{1}{V(z,t^{1/\alpha})}.$$

On the other hand,as can be seen from(7),

$$p(t,z,z) \ge \frac{c_0}{V(z,t^{1/\alpha})}.$$

So

$$|p(t,x,z)-p(t,y,z)| \le c_9\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\beta}\left(\frac{V(z,t^{1/\alpha})}{V(x,d(x,y))}\right)^{1/p}p(t,z,z);.$$

Namely

$$|p(t,x,x)-p(t,y,x)| \le c_9\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\beta}\left(\frac{V(x,t^{1/\alpha})}{V(x,d(x,y))}\right)^{1/p}p(t,x,x),$$

Further when $d(x,y) \le t^{1/\alpha}$,we know from(2)

$$|p(t,x,x)-p(t,y,x)| \le c_{10}\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\beta-(d/p)}p(t,x,x);.$$

As soon as $d(x,y) \le a t^{1/\alpha}$,for a small enough,one has

$$|p(t,x,x)-p(t,x,y)| \le \frac{1}{2}p(t,x,x),$$

So,there exists $c_{11} > 0$,such that when $y \in B(x, a t^{1/\alpha})$,there is

$$p(t,x,y) \ge \frac{1}{2}p(t,x,x) \ge \frac{c_{11}}{V(x,t^{1/\alpha})}.$$

After that,we simply call(6) $\mathrm{NLHK}(\alpha)$.To review,we say that the $\mathrm{J}_{\alpha,\le}$condition holds if there is a positive symmetric function $J(x,y)$such that for almost everywhere $x,y \in M$,

$$J(dx,dy) = J(x,y)\,\mu(dx)\,\mu(dy),$$

And there is a constant $c_1 > 0$,such that for everywhere $x,y \in M$,there is

$$J(x,y) \le \frac{c_1}{V(x,d(x,y))d^{\alpha}(x,y)}.$$

We say that the $\mathrm{CSJ}(\alpha)$condition holds if there are constants $C_0 \in (0,1]$and $C_1, C_2 > 0$ such that for any $0 < r \le R$,

$f \in \mathcal{F}$and almost everywhere $x_0 \in M$,there is a truncation function $\varphi \in \mathcal{F}_b$on $B(x_0,R) \subset B(x_0,R+r)$such that the following inequalities are true:

$$\int_{B(x_0,R+(1+C_0)r)} f^2\, d\Gamma(\varphi,\varphi) \leq C_1 \int_{U\times U^*} \left(f(x)-f(y)\right)^2 J(dx,dy) + \frac{C_2}{r^\alpha}\int_{B(x_0,R+(1+C_0)r)} f^2\, d\mu,$$

Where $\Gamma(\cdot,\cdot)$is the square field operator corresponding to $(\mathcal{E},\mathcal{F})$, $U = B(x_0,R+r) \setminus B(x_0,R)$, $U^* = B(x_0,R+(1+C_0)r) \setminus B(x_0,R-C_0 r)$and $\mathcal{F}_b = \mathcal{F} \cap L^\infty(M,\mu)$.

We say that the measurable metric space $(M,d,\mu,\mathcal{E},\mathcal{F})$satisfies the Faber-Krahn inequality $\mathrm{FK}(\alpha)$,if there are positive numbers C and νsuch that for every ball $B(x,r)$and the open set $D \subset B(x,r)$,

$$\lambda_1(D) \geq \frac{C}{r^\alpha}\left(V(x,r)/\mu(D)\right)^\nu,$$

where

$$\lambda_1(D) = \inf\{\frac{\mathcal{E}(f,f)}{|f|_2^2}: f \in \mathcal{F}\}.$$

From theorem 2.1,combined with[3,theorem 1.15],we have the following inference:

Corollary 2.2.Suppose $(\mathrm{VD})$and $(\mathrm{RVD})$are true,and if $\mathrm{FK}(\alpha)$, $\mathrm{J}_{\alpha,\leq}$and $\mathrm{CSJ}(\alpha)$are true,then $\mathrm{NLHK}(\alpha)$is true.

3. An Equivalent Characterization of Lower Bound Estimates

In this section we assume $V(x,r) \simeq r^d$,that is, $c_1,c_2>0$ exists,such that there is $c_1 r^d \leq V(x,r) \leq c_2 r^d$for any $x \in M$, $r>0$.In this case, (UHK)becomes

$$p(t,x,y) \leq C_0\left[\frac{1}{t^{d/\alpha}} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right], \tag{11}$$

And write(11)as $\mathrm{UHK}(\alpha)$.

The main conclusions of this section are:

Theorem 3.1.If the metric measure space (M,d,μ)satisfies the VD condition,and $\mathrm{UHK}(\alpha)$holds,then the following two conditions are equivalent:

(i) $\mathrm{NLHK}(\alpha)$is true,that is,there are constants $c_0,C_0>0$,for any $t>0$ and almost everywhere $x,y \in M$,when $d(x,y) \leq c_0 t^{1/\alpha}$,

$$p(t,x,y) \geq C_0 t^{-d/\alpha}. \tag{12}$$

(ii) There exist $p>1$, $\beta > d/p$, $C_1>0$,such that for any $f \in \mathcal{F}$, $x,y \in M$holds

$$\frac{|f(x)-f(y)|}{d(x,y)^{\beta-(d/p)}} \leq C_1 ||\mathcal{L}^{\beta/\alpha} f||_p. \tag{13}$$

Prove. $(ii) \Rightarrow (i)$is derived directly from theorem 2.1.Therefore,for the next major proof $(i) \Rightarrow (ii)$,we adopt the proof idea of[9,section 4].

Firstly,assuming that $\mathrm{UHK}(\alpha)$and $\mathrm{NLHK}(\alpha)$are true,then there exists $\eta \in (0,1)$such that for any $t>0$,almost everywhere

$x, y, z \in M$,there is

$$|p(t,x,z) - p(t,y,z)| \le C_1 t^{-d/\alpha}\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\eta}. \tag{14}$$

For this purpose,we use[4,Proposition 3.8].

According to[3,theorem 1.15], $\mathrm{UHK}(\alpha)$implies that $J_{\alpha,\le}$and $\mathbb{E}_{\alpha,\le}$are true if there is a constant $c_1 > 1$ such that for any $r > 0$ and $x \in M_0$,there is

$$\mathbb{E}^x[\tau_{B(x,r)}] \le c_1 r^{\alpha},$$

Among them $\tau_B(x,r) = \inf\{t > 0 : X_t \notin B(x,r)\}$.

We say that the near diagonal lower bound estimate $\mathrm{NDL}(\alpha)$of the Dirichlet heat kernel holds,if there are constants $\varepsilon \in (0,1)$ and $c_0 > 1$ such that for any $x_0 \in M, r > 0, 0 < t \le (\varepsilon r)^{\alpha}$and $B := B(x_0, r)$,there is

$$p^B(t,x,y) \ge \frac{c_1}{t^{d/\alpha}},$$

Where $x, y \in B(x_0, \varepsilon t^{1/\alpha}) \cap M_0$.The following proves that $\mathrm{NDL}(\alpha)$is true.In fact,given $x_0 \in M, R > 0$,let $B = B(x_0, R)$ .According to Hunt's formula,for any $t > 0$ and almost everywhere $x, y \in B(x_0, R/2)$,there is

$$p(t,x,y) = p^B(t,x,y) + \mathbb{E}^x\left(1_{\{t \le \tau_B\}} p(t - \tau_B, X_\tau, y)\right).$$

As can be seen from $\mathrm{UHK}(\alpha)$,there exists $c_3 > 0$ such that for any $y \in B(x_0, R/2)$,there is

$$\mathbb{E}^x\left(1_{\{t \le \tau_B\}} p(t - \tau_B, X_\tau, y)\right) \le \sup_{0 \le s \le t} \sup_{z \notin B} p(s,z,y) \le \frac{c_3 t}{R^{d+\alpha}}.$$

So

$$p^B(t,x,y) \ge p(t,x,y) - \frac{c_3 t}{R^{d+\alpha}}.$$

Next,let $x, y \in B(x_0, \delta R), t = \theta R^{\alpha}$where $\theta = (C_0/2c_3)^{\alpha/(d+\alpha)}$and $\delta \le 1/2$ satisfies $2\delta R \le c_0 t^{1/\alpha}$,where c_0, C_0comes from(12).From $\mathrm{NLHK}(\alpha)$,it can be seen that

$$p(t,x,y) \ge \frac{C_0}{t^{d/\alpha}},$$

Further,according to the selection of θ,we know that

$$p^B(t,x,y) \ge \frac{C_0}{t^{d/\alpha}} - \frac{c_3 t}{R^{d+\alpha}} \ge \frac{C_0}{2t^{d/\alpha}},$$

Thus, $\mathrm{NDL}(\alpha)$holds.Thus,it can be inferred from[4,Proposition 3.8]that(14)holds

Next,we prove,for any $1 \le p < +\infty, t > 0$ and $f \in L^p(M, \mu)$and almost everywhere $x, y \in M$,there is

$$\left|e^{-t\mathcal{L}} f(x) - e^{-t\mathcal{L}} f(y)\right| \le C_1 d(x,y)^{\eta/p} t^{-\gamma/\alpha} ||f||_p, \tag{15}$$

Where $\gamma = (d + \eta)/p$.In fact,for $p, p' > 1$ satisfies $1/p' + 1/p = 1$,there is

$$\left|e^{-t\mathcal{L}} f(x) - e^{-t\mathcal{L}} f(y)\right| = \left|\int_M \left(p(t,x,z) - p(t,y,z)\right) f(z)\, \mu(dz)\right|$$

$$\le ||p(t,x,\cdot) - p(t,y,\cdot)||_{p'} ||f||_p.$$

Further,it can be seen from(14)

$$||p(t,x,\cdot) - p(t,y,\cdot)||_{\infty} \le C_1 t^{-d/\alpha}\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\eta};$$

On the other hand,by the Markov property of $e^{-t\mathcal{L}}$,there are

$$||p(t,x,\cdot)-p(t,y,\cdot)||_1 \le 2.$$

From the inequality Hölder,we know

$$||p(t,x,\cdot)-p(t,y,\cdot)||_{p'} \le C_1 t^{-d/\alpha p}\left(\frac{d(x,y)}{t^{1/\alpha}}\right)^{\eta/p},$$

Then(15)be proved.

Now,take $k>\gamma/\alpha$and $f\in L^p(M,\mu)$,because

$$f=c(k)\int_0^{+\infty} t^{k-1}\mathcal{L}^k e^{-t\mathcal{L}} f\,dt,$$

So

$$|f(x)-f(y)| \le c(k)\int_0^{+\infty} t^{k-1}\left|\mathcal{L}^k e^{-t\mathcal{L}} f(x)-\mathcal{L}^k e^{-t\mathcal{L}} f(y)\right|\,dt$$

$$=c(k)\int_0^{+\infty} t^{k-1}\left|e^{-(t/2)\mathcal{L}}\mathcal{L}^k e^{-(t/2)\mathcal{L}} f(x)-e^{-(t/2)\mathcal{L}}\mathcal{L}^k e^{-(t/2)\mathcal{L}} f(y)\right|\,dt.$$

Finally,it can be inferred from(15)that

$$|f(x)-f(y)| \le C_1 d(x,y)^{\eta/p}\int_0^{+\infty} t^{k-\gamma/\alpha-1}\,||\mathcal{L}^k e^{-(t/2)\mathcal{L}} f||_p\,dt.$$

By using the difference theorem(see[9],Proposition 4.3]for details),the desired conclusion (ii)can be proven to hold.

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