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Hardy's Inequality on Metric Measure Spaces and Its Characterization

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Abstract: We consider the Hardy inequality on metric measure spaces and establish the characterization of the remainder term in the Hardy inequality. In particular, the probabilistic meaning of the remainder is explicitly presented.
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1. Introduction

The Hardy inequality has a long history of research, which originates from Hardy's work in 1925 (see ref. [6]). Hardy inequality has been widely used in many fields such as mathematics and physics, especially in harmonic analysis, functional analysis, partial differential equations and so on. For example, in ref. [7], Maz'ya et al used Hardy inequality to obtain results of priori estimation, existence and regularity. In ref, Vázque et al. used Hardy inequality to study the well-posedness and asymptotic behavior of solutions of some partial differential equations. Recently, Cao Jun et al., in ref. [3], proved the abstract form of local and non-local regular Dirichlet Hardy inequality on the metric measure space by using the Green operator of Dirichlet type. The purpose of this paper is to give Hardy inequality with remainders in metric measure space, and to give the fine characterization and analysis of the remainders.

2. Main Conclusions and Proofs

To illustrate our main conclusions, the main signs are described as follows. (M,d,μ) is noted as the metric measure space where μ is the Radon measure on M and its corresponding support is the total space M. $L^2(M,\mu)$ is used to represent the total square integrable measurable function on M, and its corresponding norm is denoted as $\| f\|_2$. Let $(\mathcal{E},\mathcal{F})$ be the regular symmetric Dirichlet form of $L^2(M,\mu)$, and define $\{P_t\}_{t\geq 0}$and $\mathcal{L}$ as their corresponding Markov semigroups and generator operators respectively, meanwhile, $P_t(x,dy)$ is noted as the Markov transfer kernel corresponding to the semigroup P_t. please note that we do not require $P_t(x,dy)$ to be conservative here. The main theorems and proofs of this paper are given below.

Theorem 2.1 For any $0 < h \in \mathcal{F}, f \in \mathcal{F}$ satisfies $h^{-1}f^2 \in$, there is

$$\mathcal{E}(f,f) = \mathcal{E}(h,h^{-1}f^2) + \lim_{t\to 0}\frac{1}{2t}\int_M\int_M h(x)h(y)\left(h(x)^{-1}f(x) - h(y)^{-1}f(y)\right)^2 P_t(x,dy)\,\mu(dx) \tag{2.1}$$

Proof: For any $t > 0$, define

$$\mathcal{E}_t(u,u) = \frac{1}{t}\int_M \left(u(x) - P_t u(x)\right)u(x)\ \mu(dx),\ u \in \mathcal{F}$$

According to ref. [5, Lemma 1.3.4], it can be seen that for any $u \in \mathcal{F}$, satisfies

$$\lim_{t\to 0}\mathcal{E}_t(u,u) = \mathcal{E}(u,u) \tag{2.2}$$

Further, we have

$$\mathcal{E}_t(u,u) = \frac{1}{t}\int_M \big(u(x) - P_t u(x)\big)u(x)\ \mu(dx)$$

$$= \frac{1}{t}\int_M \left(u(x) - \int_M u(y)P_t(x,dy)\right)u(x)\ \mu(dx)$$

$$= \frac{1}{t}\int_M\int_M \big(u(x) - u(y)\big)P_t(x,dy)u(x)\ \mu(dx) + \frac{1}{t}\int_M\int_M u(x)^2\big(1 - P_t(x,dy)\big)\ \mu(dx)$$

$$= \frac{1}{t}\int_M\int_M \big(u(x) - u(y)\big)P_t(x,dy)u(x)\ \mu(dx) + \frac{1}{t}\int_M u(x)^2\big(1 - P_t(x,M)\big)\ \mu(dx)$$

According to the symmetry of Markov semigroups $\{P_t\}_{t\geq 0}$,

$$\mathcal{E}_t(u,u) = \frac{1}{2t}\int_M\int_M \big(u(x) - u(y)\big)^2 P_t(x,dy)\ \mu(dx) + \frac{1}{t}\int_M u(x)^2\big(1 - P_t(x,M)\big)\ \mu(dx) \tag{2.3}$$

Let's prove the conclusion of equation (2.1) below. In fact, for any $0 < h \in \mathcal{F}, f \in \mathcal{F}$ and $h^{-1}f^2 \in \mathcal{F}$, According to equation (2.3),

$$\mathcal{E}_t(h, h^{-1}f^2) = \frac{1}{2t}\int_M\int_M \big(h(x) - h(y)\big)(h(x)^{-1}f(x)^2 - h(y)^{-1}f(y)^2)P_t(x,dy)\ \ \mu(dx) + \frac{1}{t}\int_M h(x)h(x)^{-1} f(x)^2(1 - P_t(x,M))\ \mu(dx)$$

$$= \frac{1}{2t}\int_M\int_M (h(x) - h(y))(h(x)^{-1}f(x)^2 - h(y)^{-1}f(y)^2)P_t(x,dy)\mu(dx) + \frac{1}{t}\int_M f(x)^2(1 - P_t(x,M))\ \mu(dx)$$

Note that,

$$\begin{aligned}
&\big(h(x) - h(y)\big)(h(x)^{-1}f(x)^2 - h(y)^{-1}f(y)^2)\\
&= f(x)^2 + f(y)^2 - h(x)h(y)^{-1}f(y)^2 - h(y)h(x)^{-1}f(x)^2\\
&= \big(f(x) - f(y)\big)^2 + 2f(x)f(y) - h(x)h(y)\big(h(y)^{-1}f(y)\big)^2 - h(y)h(x)\big(h(x)^{-1}f(x)\big)^2\\
&= \big(f(x) - f(y)\big)^2 + h(x)h(y)(2h(x)^{-1}f(x)h(y)^{-1}f(y) - \big(h(y)^{-1}f(y)\big)^2 - \big(h(x)^{-1}f(x)\big)^2\big)\\
&= \big(f(x) - f(y)\big)^2 - h(x)h(y)\big(h(x)^{-1}f(x) - h(y)^{-1}f(y)\big)^2
\end{aligned}$$

So,

$$\mathcal{E}_t(f,f) - \mathcal{E}_t(h, h^{-1}f^2) = \frac{1}{2t}\int_M\int_M h(x)h(y)\big(h(x)^{-1}f(x) - h(y)^{-1}f(y)\big)^2 P_t(x,dy)\ \mu(dx)$$

According to equation (2.2), let $t \to 0$, to obtain our conclusion.

Note that, when $h \in \mathcal{F}$,

$$\mathcal{E}(h, h^{-1}f^2) = \langle -\mathcal{L}h, h^{-1}f^2\rangle_{L^2(\mu)} = -\int_M f^2\frac{\mathcal{L}h}{h}d\mu$$

So, we can infer as follows.

Corollary 2.2 when $0 < h \in \mathcal{F}, f \in \mathcal{F}$,

$$\mathcal{E}(f,f) \geq -\int_M f^2 \frac{\mathcal{L}h}{h} d\mu$$

The above corollary is consistent with the conclusion obtained in ref. [4]. In fact, under the condition that $\mathcal{L}$ is a symmetric generator of the Dirichlet type $\mathcal{E}$ and h is an Superharmonic function, i.e., $h \geq 0$ and $\mathcal{L}h \leq 0$, ref. [4] proves

$$\mathcal{E}(f,f) \geq \int f^2 \left(\frac{-\mathcal{L}h}{h}\right)$$

is valid. However, in Corollary 2.2, we do not require that h be an up-sum function, i.e., $\mathcal{L}h \leq 0$. There are many profound results on the Dirichlet Hardy inequality, for example, see ref. [3,1].

In addition, theorem 2.1 gives a characterization of the Hardy inequality with remainders, and in fact the Hardy inequality with remainders has been given in ref. [2]. Specifically, if $u \in \mathcal{F}$, $u = 0$ is satisfied on the set $\{x \in M{:}h(x) = 0, h = \infty\}$, then there is

$$\mathcal{E}(u,u) \geq \int_M u(x)^2 q(x) \mu(dx)$$
$$+ \liminf_{t\to 0} \int_M \int_M \frac{p_t(x,y)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)}\right)^2 h(y)h(x) \mu(dy) \mu(dx)$$

Here $p_t(x,y)$ is the thermonuclear corresponding to Dirichlet type $(\mathcal{E}, \mathcal{F})$. The proof of the above conclusion makes full use of the non-explosive property of the perturbation of Schrödinger semi-group.

3. Analysis of Remainders of Hardy Inequality

In this section, we define

$$R(f,f) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M h(x)h(y)\left(h(x)^{-1}f(x) - h(y)^{-1}f(y)\right)^2 P_t(x,dy) \mu(dx) \tag{3.1}$$

Next, we will analyze the probabilistic meaning of $R(f,f)$, for which the constraint function h, of theorem 1.1, is defined

$$\mathcal{L}_h f(x) = h(x)^{-1}\mathcal{L}(hf)(x) - h(x)^{-1}\mathcal{L}h(x) \cdot f(x)$$
$$=: \mathcal{L}_h^* f(x) - h(x)^{-1}\mathcal{L}h(x) \cdot f(x) \tag{3.2}$$

We first have the following lemma.

Lemma 3.1 note $\mu_h(dx) = h(x)^2\mu(dx)$, then $\mathcal{L}_h$ is a symmetric operator in the space $L^2(M, \mu_h)$.

Proof: For any $f,g \in L^2(M, \mu_h)$, there is

$$\langle f, \mathcal{L}_h^* g\rangle_{\mu_h} = \left\langle h^{-1}, f\mathcal{L}(hg)\right\rangle_{\mu_h} = \langle hf, \mathcal{L}(hg)\rangle_\mu$$
$$= \langle \mathcal{L}(hf), hg\rangle_\mu = \left\langle h^{-1}\mathcal{L}(hf), g\right\rangle_{\mu_h}$$
$$= \langle \mathcal{L}_h^* f, g\rangle_{\mu_h}$$

Then $\mathcal{L}_h^*$ is symmetric in the space $L^2(M, \mu_h)$. Thus, according to the definition of $\mathcal{L}_h$, the desired conclusion can be obtained.

Now let's consider the expression $\langle \mathcal{L}_h, f\rangle_{\mu_h}$. Take note of,

$$\mathcal{L}f(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}$$
$$= \lim_{t\to 0} \left[\frac{1}{t} \int_M \left(f(y) - f(x)\right) P_t(x,dy) - \frac{1}{t}\left(1 - P_t(x,M)\right) f(x) \right]$$

Consequently

$$\langle \mathcal{L}_h f, f\rangle_{\mu_h} = \langle hf, \mathcal{L}hf\rangle_\mu - \int_M h(x)\mathcal{L}h(x)\cdot f(x)^2\,\mu(dx)$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\int_M h(x)f(x)\int_M \big(h(y)f(y) - f(x)h(x)\big)P_t(x,dy)\ \mu(dx) - \frac{1}{t}\int_M \big(1 - P_t(x,M)\big)h(x)^2 f(x)^2\,\mu(dx)\right]$$

$$- \lim_{t\to 0}\left[\frac{1}{t}\int_M h(x)f(x)^2\int_M \big(h(y) - h(x)\big)P_t(x,dy)\ \mu(dx) - \frac{1}{t}\int_M \big(1 - P_t(x,M)\big)h(x)^2 f(x)^2\,\mu(dx)\right]$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\int_M\int_M h(x)f(x)\big(h(y)f(y) - f(x)h(x)\big)P_t(x,dy)\ \mu(dx) - \frac{1}{t}\int_M h(x)f(x)^2\int_M \big(h(y) - h(x)\big)P_t(x,dy)\ \mu(dx)\right]$$

$$= \lim_{t\to 0}\left[\frac{1}{t}\int_M\int_M h(x)h(y)(f(x)f(y) - f(x)^2)P_t(x,dy)\ \mu(dx)\right]$$

$$= -\lim_{t\to 0}\left[\frac{1}{2t}\int_M\int_M h(x)h(y)\big(f(x) - f(y)\big)^2\ P_t(x,dy)\ \mu(dx)\right]$$

The last equation can be obtained from the symmetry of the transfer kernel.

In summary, we have the following proposition.

Proposition 3.2 for any $0 < h \in \mathcal{F}$ and $f \in \mathcal{F}$, there is

$$R(f,f) = -\left\langle \mathcal{L}_h\frac{f}{h}, \frac{f}{h}\right\rangle_{\mu_h}$$

Among them, $\mathcal{L}_h$ is given by equation (3.2). Specially, $f \mapsto -\langle \mathcal{L}_h f, f\rangle$ is non negative quadratic form on $L^2(M,\mu_h)$.

Note 3.3 In this article we do not assume that $P_t(x,dy)$ is conservative, roughly speaking, we do not assume that $\mathcal{L}1$ is equal to 0. It can be seen from equation (3.2) that $\mathcal{L}_h$ satisfies $\mathcal{L}_h 1 = 0$, so roughly speaking, $\mathcal{L}_h$ will correspond to the conservative Markov process. On the other hand, in fact, $\mathcal{L}_h$has been used to characterize the first eigenvalue of the Markov process in article , but it is not proved that $-\mathcal{L}_h$ is non-negative definite on the space $L^2(M,\mu_h)$.

Combining Theorem 2.1 and proposition 3.2, we get,

Corollary 3.4 for any $0 < h \in \mathcal{F}$ and $f \in \mathcal{F}$, there is

$$\mathcal{E}(f,f) = \int_M \frac{-\mathcal{L}h}{h} f^2\,d\mu + \left\langle -\mathcal{L}_h\frac{f}{h}, \frac{f}{h}\right\rangle_{\mu_h}$$

Where $\mathcal{L}_h$ is given by equation (3.2).

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