DOI:10.18686/ahe.v8i4.13344

Mixed Variables Diophantine Inequality with Prime Variables

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Abstract: Let $1 < k < 8/3$, λ_1, λ_2, λ_3 and λ_4 be non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let η be a real number. The inequality $\left|\lambda_1 P_1 + \lambda_2 P_2^2 + \lambda_3 P_3^2 + \lambda_4 P_4^k + \eta\right| \leq \left(maxP_j\right)^{-\sigma+\varepsilon}$ has infinitely many solutions in prime variables p_1, p_2, p_3 and p_4 for any $\varepsilon > 0$.
Keywords: Diophantine inequalities; Hardy-Littlewood method; Davenport-Heilbronn method

1. Proof method

Where k and η is any given real number, ε and δ are given positive numbers which are small enough. The letter P represents a prime number, let q be a denominator of a convergent to λ_1/λ_2.

Set $e(\alpha) = e^{2\pi i\alpha}$, $X = q^2$, $j \geq 1$,

$$S_j(\alpha) = \sum_{\delta X \leq p^i \leq X} (logp)e(p^i\alpha) \quad U_j(\alpha) = \sum_{\delta X \leq n^i \leq X} e(n^i\alpha) \quad T_j(\alpha) = \int_{(\delta X)^{1/j}}^{X^{1/j}} e(\alpha t^i)dt$$

By prime number theorem and Trigonometric integral, we have

$$S_j(\alpha) \ll X^{1/j}, \quad T_j(\alpha) \ll X^{1/j} min(X, |\alpha|^{-1}). \tag{1}$$

$$if\ \alpha \neq 0, K_\tau(\alpha) = \left(\frac{sin\pi\tau\alpha}{\alpha\pi}\right)^2; if \alpha = 0, K_\tau(\alpha) = \tau^2, \quad \text{So} \quad K_\tau(\alpha) \ll min(\tau^2, |\alpha|^{-2}) \tag{2}$$

$$K_\tau(\alpha) \ll min(\tau^2, |\alpha|^{-2}), \tag{3}$$

$$\int e(xy) K_\tau(x)dx = max(0, \tau - |y|), \tau > 0. \tag{4}$$

$$U_k(\alpha) - T_k(\alpha) << 1 + |\alpha|X \tag{5}$$

$$J_r(X,h) = \int_X^{2X} \left[\left(\theta\left((x+h)^{\frac{1}{k}}\right) - \theta\left(x^{\frac{1}{k}}\right) - \left((x+h)^{\frac{1}{k}}\right) - x^{\frac{1}{k}}\right)\right]^2 dx$$

Which $\theta(x) = \sum_{p \leq x} logp$ is Chebyshev function.

Lemma 1[1]: let $k \geq 1$, $\int_{-Y}^{Y} |S_r(\alpha) - U_r(\alpha)|^2 d\alpha \ll_r \frac{X^{\frac{2}{r}-2} log^2 X}{Y} + Y^2 X + Y^2 J_r\left(X, \frac{1}{2Y}\right)$.

Lemma 2[1]: let $k \geq 1$, $J_r(X,h) \ll_r h^2 X^{\frac{2}{r}-1} exp\left(-c_1\left(\frac{logX}{loglogX}\right)^{\frac{1}{3}}\right), X^{1-\frac{5}{6r}+\varepsilon} \ll h \ll X$.

Inference 1: let $r \geq 1$ and any $A \geq 6$,

$$\int_{|\alpha| \leq X^{-1+\frac{5}{6r}-\varepsilon}} \left|S_r(\lambda_j\alpha) - U_r(\lambda_j\alpha)\right|^2 d\alpha < X^{\left(\frac{2}{r}\right)-1}(logX)^{-A}.$$

Lemma 3[2] : for $j = 1,2$, $\int_m \left|S_j(\lambda_j\alpha)\right|^{2j} |K_\tau(\alpha)| d\alpha << \tau X(logX)^j$,

$$\int_m |K_\tau(\alpha)|\, |S_k(\lambda_4)|^4 d\alpha << \tau X^{1/k} X^{1/k} (logX)^3.$$

Lemma 4 [3]: $\int_{|\alpha|\le X^{-2/3}} |S_2(\lambda_j\alpha)|^2 d\alpha << 1.$

Let: $R = \{(p_1, p_2, p_3, p_4) | \delta X \le p_1 \le X, \delta X \le p_2^2, p_3^2 \le X, \delta X \le p_4^k \le X\}$,define:

$G(\alpha) = S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\tau(\alpha)e(\alpha\eta)$, $H(\alpha) = T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\tau(\alpha)e(\alpha\eta)$.

As for any measurable subset Ω of Lebesgue of real number set R , $I(\tau, \eta, \Omega) = \int_\Omega G(\alpha)\, d\alpha$, $(\tau, \eta, R) \le \tau(\log X)^4 \Im(x)$.Where $\Im(x)$ represents the number of solutions in the inequality

$$|\lambda_1 p_1{}^1 + \lambda_2 p_2{}^2 + \lambda_3 p_3{}^2 + \lambda_4 p_4{}^k + \eta| < \tau \quad (6)$$

Let $R = M \cup m \cup t$ where M is the major arc, m is minor arc and T is the trivial arc. The decomposition is the following:

$M = \{\alpha | |\alpha| \le P/X\}$, $m = \{\alpha | P/X < |\alpha| < Q\}$, $T = \{\alpha | |\alpha| \ge Q\}$

So that $I(\tau, \eta, M) = I(\tau, \eta, M) + I(\tau, \eta, m) + I(\tau, \eta, T)$.

2. The major mac

$$I(\tau, \eta, M) = \int_M (T_1(\lambda_1\alpha))\, T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\tau(\alpha)e(\eta\alpha))d\alpha$$

$$+ \int_M (S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha))T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha$$

$$+ \int_M S_1(\lambda_1\alpha)(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha))\, T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha$$

$$+ \int_M S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\tau(\alpha)e(\eta\alpha)d\alpha$$

$$+ \int_M S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)(S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha))K_\tau(\alpha)e(\eta\alpha)d\alpha$$

$$=: C_1 + C_2 + C_3 + C_4 + C_5$$

2.1 Lower bound for C_1.

Using inequalities (2) and (4)

$$C_1 = \int_M H(\alpha)\, d\alpha + O(\int_{P/X}^{+\infty} |H(\alpha)|\, d\alpha)$$

$$\int_{P/X}^{+\infty} |H(\alpha)| d\alpha << \tau^2 X^{\frac{1}{k}-1} \int_{P/X}^{+\infty} \alpha^{-4} d\alpha << \tau^2 P^{-3} X^{\frac{1}{k}+1} = \tau^2 X^{\frac{1}{k}+1} P^{-3} = o\left(\tau^2 X^{\frac{1}{k}+1}\right)$$

let$D = [\delta X, X] \times [(\delta X)^{1/2}, X^{1/2}]^2 \times [(\delta X)^{1/k}, X^{1/k}]$, we have$\int_R H(\alpha) d\alpha \gg \tau^2 X^{\frac{1}{k}+1}$,so $C_1 \gg \tau^2 X^{1+(1/k)}$.

In fact, since the computations for C_2, C_3, C_4 and C_5 similar to, but simpler than, so you will get the corresponding ones for C_3.

2.2 Bound for C_3

$$C_3 \ll \tau^2 \int_M S_1(\lambda_1\alpha)(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha))\, T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)d\alpha$$

$$\ll \tau^2 \int_M S_1(\lambda_1\alpha)|S_2(\lambda_2\alpha) - U_2(\lambda_2\alpha)|\, T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)d\alpha$$

$$+ \tau^2 \int_M S_1(\lambda_1\alpha)|U_2(\lambda_2\alpha) - T_2(\lambda_2\alpha)|\, T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)d\alpha \; =: \tau^2(C_{31} + C_{32}).$$

$$C_{31} \ll X^{1/2} X^{1/k} \left(\int_M |S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha)|^2 d\alpha\right)^{1/2} \left(\int_M |S_1(\lambda_1\alpha)|^2 d\alpha\right)^{1/2}$$

$$\ll X^{\frac{1}{2}+\frac{1}{k}}((\log X)^{-A})^{\frac{1}{2}}(X\log X)^{\frac{1}{2}} \ll X^{1+\frac{1}{k}}(\log X)^{\frac{-A}{2}} \ll X^{1+\frac{1}{k}}$$

$$C_{32} \ll \int_0^{1/X} |S_1(\lambda_1\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)| d\alpha + \int_{1/X}^{P/X} \alpha X\, |S_1(\lambda_1\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)| d\alpha$$

$$\ll (X\log X \cdot X\log^2 X)^{1/2} + P(X\log X \cdot X\log^2 X)^{1/2} = (1 + P)(X\log X \cdot X\log^2 X)^{1/2}$$

Since we need $C_{32} = o(X^{1+1/k})$, so that $P = o(X^{1/k})$ .Calculate according to this method, we need

$C_2 = C_4 = C_5 = o\left(X^{1+\frac{1}{k}}\right)$, so that $P = o\left(X^{\frac{2}{5}}, X^{\frac{5}{6k}}\right)$.

Collecting all the bounds of P , if $1 \le k \le 25/12, P = X^{2/5}$, else $P = 5/6k$

3. The trivial arc

By the airthmetic-geometric mean inequality and $S_k(\lambda_4\alpha) \ll X^{1/k}$, we see that

$$I(\tau,\eta,t) \ll X^{\frac{1}{k}} \int_R^{+\infty} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)|K_\tau(\alpha)\,d\alpha$$

$$\ll X^{\frac{1}{k}}\left(\int_R^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\,d\alpha\right)^{1/2} \left(\int_R^{+\infty} |S_2(\lambda_2\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1/4}$$

$$\times \left(\int_R^{+\infty} |S_2(\lambda_3\alpha)|^4 K_\tau(\alpha)\,d\alpha\right)^{1/4} \ll X^{\frac{1}{k}}\left(\frac{(X\log X)}{Q}\right)^{1/2}\left(\frac{(X\log X)}{Q}\right)^{1/2} \ll \frac{X^{1+(1/k)+\varepsilon}}{Q}.$$

Remembering that $I(\tau,\eta,t) = o\left(\tau^2 X^{1+(1/k)+\varepsilon}\right)$, i.e. of the main term, the choice $Q = log^2 X/\tau^2$ is admissible.

4. The minor arc

Let us split m into two subsets m^* and $\tilde{m} = m \setminus m^*$, $\tilde{m} = m_1 \cup m_2$, where

$$m_1 = \{\alpha \in m: |S_1(\lambda_1\alpha)| < X^{1-1/8+\varepsilon}\}$$

$$m_2 = \{\alpha \in m: |S_2(\lambda_2\alpha)| < X^{1/2-1/16+\varepsilon}\}.$$

Using the Holder inequality ,lemma3 and Theorem3.1in [4] we obtain,

$$|I(\tau,\eta,m_1)| \ll max S_1(\lambda_1\alpha)^{\frac{1}{2}}\left(\int_{m_1} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)d\alpha\right)^{1/2}\left(\int_{m_1} |S_2(\lambda_2\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1/4}$$

$$\times \left(\int_{m_1} |S_2(\lambda_3\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1/4}\left(\int_{m_1} |S_k(\lambda_4\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1}$$

$$\ll X^{\frac{7}{16}}(\tau X\, log^2 X)^{\frac{1}{2}}\left(\tau X^{\frac{1}{k}} X^{\frac{2}{k}} log^3 X\right)^{1/4} \ll \tau X^{\frac{19}{16}+\frac{1}{2k}+\varepsilon} logX = o(\tau^2 X^{1+\frac{1}{k}})$$

So$\tau = X^{-(8-3k)/16k+\varepsilon}$.

Let $|S_1(\lambda_1\alpha)| > X^{1-(1/8)+\varepsilon}$, $|S_2(\lambda_2\alpha)| > X^{(1/2)-(1/16)+\varepsilon}$, $P/X < |\alpha| \le \tau^{-2} log^2 X$. Following the method in [5] we divide m^* into disjoint set $E(Z_1, Z_2, y)$ in which ,for $\alpha \in E(Z_1, Z_2, y)$, we have $Z_1 < |S_2(\lambda_1\alpha)| \le 2Z_1, Z_2 < |S_2(\lambda_2\alpha)| \le 2Z_2, y < |\alpha| \le 2y$. Where $Z_1 = 2^{k_1} X^{1-(1/8)+\varepsilon}, Z_2 = 2^{k_2} X^{(1/2)-(1/16)+\varepsilon}, y_1 = 2^{k_3} X^{-3/5+\varepsilon}, y_2 = 2^{k_4} X^{(5/6k)-1+\varepsilon}$. k_1, k_2, k_3, k_4 are non-negative integer.

we have the following result about $\mu(A)$ following the same lines of lemma 4 in [2].

Lemma 5: $\mu(A) \ll y X^{2+6\varepsilon} Z_1^{-2} Z_2^{-4}$.

$$I(\tau,\eta,A) \ll \int_{m^*} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)||K_\tau(\alpha)|d\alpha$$

$$\ll \left(\int_{m^*} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1/4}\left(\int_{m^*} |S_2(\lambda_3\alpha)|^4 K_\tau(\alpha)d\alpha\right)^{1/4}$$

$$\times \left(\int_{m^*} |S_k(\lambda_4\alpha)|^2 K_\tau(\alpha)d\alpha\right)^{1/2} \ll \left(min\,(\tau^2, y^{-2})\right)^{\frac{1}{4}}\left((Z_1 Z_2)^4 \mu(A)\right)^{\frac{1}{4}}(\tau X log^2 X)^{\frac{1}{4}}\left(\tau X^{\frac{1}{k}+\varepsilon}\right)^{\frac{1}{2}}$$

$$\ll \left(min\,(\tau^2, y^{-2})\right)^{\frac{1}{4}} y^{\frac{1}{4}} Z_1^{\frac{1}{2}} X^{\left(\frac{5}{8}\right)+\left(\frac{3\varepsilon}{2}\right)} \tau^{\frac{3}{4}} X^{\left(\frac{1}{3}\right)+\left(\frac{1}{2k}\right)}$$

$If\; y_1 = 2^{k_3} X^{-\frac{3}{5}+\varepsilon}, I(\tau,\eta,A) \ll \tau X^{\frac{87}{80}+\frac{1}{2k}}; y_2 = 2^{k_4} X^{(5/6k)-1+\varepsilon}, I(\tau,\eta,A) \ll \tau X^{\frac{19}{16}+\frac{5}{24k}}$. So $\tau = X^{-\frac{(8-3k)}{16k}+\varepsilon}$ is optimal choice.

References:

[1]Alessandro Gambini. Diophantine approximation with one prime,two squares of primes and one k-th power of a prime. https://arxiv.org/abs/1706.00343 .

[2] Quanwu Mu , Xiaodong Lv. Diophantime Approximation With Prime Variable and Mixed powers(II)[J].Advances in Mathematic (China), 2017, 46(2): 190-202.

[3]李伟平, 王天泽等. 素变量混合幂丢番图逼近[J]. 数学学报中文版,2019,62(1): 50-58.

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