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Finite-Dimensional Hopf Algebras

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Abstract: This paper mainly discussed various characterizations for the finite-dimensional Hopf algebras over algebraically closed field and has characteristic 0. And further showed that the order of antipode of the Hopf algebras is finite, but also provides a hint on how to estimate the order of the antipodes.

Keywords: Finite-dimensional Hopf algebras; Order of the antipode ; Trace; Semisimple; Cosemisimple; Eigenvalue; Distinguished grouplike

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1. Introduction

Throughout this paper K is a algebraically closed field and has characteristic0, H is a finite -dimensional $K-Hopf$ algebra with antipode S which is a diagonalizable operator and C is a $K-$

coalgebra . There is a convenient adaptation of the Heyneman–Sweedler[1] singma notation for coalgebras and comodules as $\Delta(c)=\sum c_{(1)}\otimes c_{(2)}$ and $\rho(c)=\sum c_{(-1)}\otimes c_{(0)}$ $\forall c\in C$.

Definition1.1[1] A grouplike elements of C is a $c\in C$ which satisfies the following conditions: $\Delta(c)=c\otimes c$ and $\varepsilon(c)=1$, the set of grouplike elements of C is denoted $G(C)$.

We firstly recall the following actions as module structures:

H^* is a left $H-$module via $(h\rightarrow h^*)(g)=h^*(gh)$ for $h,g\in H,h^*\in H^*$.

H^* is a rigt $H-$module via $(h\leftarrow h^*)(g)=h^*(hg)$ for $h,g\in H,h^*\in H^*$.

H is a left hH^*-module via $h^*\rightarrow h=\sum h^*(h_2)h_1$ for $h^*\in H^*,h\in H$.

H is a right H^*-module via $h\leftarrow h^*=\sum h^*(h_1)h_2$ for $h^*\in H^*,h\in H$.

If $g\in H$ is a grouplike element as in Definition1.1, we can denote by

$$L_g=\{m\in H^*\,|\,h^*m=h^*(g)m \text{ for any } h^*\in H^*\}$$

and

$$R_g=\{n\in H^*\,|\,nh^*=h^*(g)n \text{ for any } h^*\in H^*\}$$

which are ideals of H^* and $L_1=\int_l$, $R_1=\int_r$. Also recall from[1] that L_g and R_g are 1-dimensional,and there exists a grouplike element d such that $R_d=L_1$, where d is called the distinguished grouplike. We can perform the same constructions on the dual algebra H^*. More precisely, for any $\eta\in G(H^*)=A\lg(H,K)$ we can define

$$L_\eta=\{x\in H\,|\,hx=\eta(h)x \text{ for any } h\in H\}$$
$$R_\eta=\{y\in H\,|\,yh=\eta(h)y \text{ for any } h\in H\}.$$

We remark that if we keep the same definition we gave for L_g, then L_η should be a subspace

of H^{**}.The set L_η, as defined above ,is just the preimage of this subspace via the canonical

Isomorphism $\theta:H\rightarrow H^{**}$. From the above it follows that the subspaces L_η and R_η are ideals

of dimension 1 in H, and there exists $\alpha\in G(H^*)$ such that $R_\alpha=L_\varepsilon$. This element α is the

distinguished grouplike element in H^*.

Remark1.2[1] If H is semisimple and cosemisimple, then distinguished grouplike in H and

H^* are equal to 1 and ε, respectively.

Lemma1.3[2] Suppose that H is a Hopf algebra over K. Then

The only subspaces of H which are both a left ideal and left coideal of H are $\{O\}$ and

H.

If H contains a non-zero finite -dimensional left or right ideal. Then H is finite-dimensional.

Lemma1.4[3] Let C be a finite-dimensional coalgebra over K. Then $U \mapsto U^{\perp}$ is a one-one inclusion reversing correspondence between the set of coideals (respectively subcoalgebras, left coideals, right coideals) of C and the set of subalgebras (respectively ideals, left ideals, right ideals) of the dual algebra C^{*}.

Lemma1.5[3] If $C_n(K)$ is a simple coalgebra over K for all $n \geq 1$. Then any simple coalgebra over K is isomorphic to $C_n(K)$ for some $n \geq 1$.

Lemma1.6[4] Suppose U and V be vector spaces over K and $F: V^{*} \to U^{*}$ is the transpose of a linear map $f: U \to V$. If J and I are subspaces of V^{*} and U^{*} respectively. Then

$F(J) \subseteq I$ implies $f(I^{\perp}) \subseteq J^{\perp}$.

Remark1.7[4] For a subspace V of U let $res_V^U : U^{*} \to V^{*}$ be the restriction map which is thus defined by $res_V^U(u^{*}) = u^{*}|V$ for all $u^{*} \in U^{*}$. Notice that $Ker(res_V^U) = V^{\perp}$. Hence

$U^{*}/V^{\perp} \cong V^{*}$ as vector spaces. Therefore we have the formula $Dim(U^{*}/V^{\perp}) = Dim(V^{*})$ In particular $V^{\perp}$ is a cofinite subspace of U^{*} if and only if V is a finite-dimensional subspace of

U. Also notice that $res_V^U = i^{*}$, where $i: V \to U$ is the inclusion map.

Definition1.8[4] For $a \in H$, $a^{*} \in H^{*}$, $b \in H$, define endomorphisms $L(a^{*})$ and $R(a^{*})$ in $End(H)$ by $L(a^{*})(b) = a^{*} \to b$ and $R(a^{*})(b) = b \leftarrow a^{*}$, on the other hand, $l(a)$ and $r(a)$ in $End(H)$ by $l(a)(b) = ab$ and $r(a)(b) = ba$.

Proposition 1.9[5] Suppose that S is the antipode of H. Let Λ be a left integral for H and ω be a right integral for H^{*} which satisfy $<\Lambda, \omega> = 1$. Then

$Tr(r(a) \circ S^{2} \circ R(a^{*})) = <\omega, a><a^{*}, \Lambda>$ for all $a \in H$, $a^{*} \in H^{*}$.

The functional $\omega_r \in H^{*}$ defined by $\omega_r(a) = Tr(r(a) \circ S^{2})$ for all $a \in H$ is a right integral for H^{*}.

Proposition 1.10[5] Suppose that S is the antipode of H. Then the following are equivalent:

H and H^{*} are semisimple.

$Tr(S^{2}) \neq 0$.

Proposition 1.11[5] Suppose that S is the antipode of H.

Let g and α be the distinguished grouplike elements for H and H^{*} respectively. Then $S^{4} = \tau_g \circ (\tau_{\alpha^{-1}})^{*}$

or equivalently, $S^{4}(a) = g(\alpha \to a \leftarrow \alpha^{-1})g^{-1}$, for all $a \in H$.

If H and H^{*} are unimodular, in particular if H and H^{*} are semisimple, then $S^{4} = 1_H$.

$Tr(S^{2}) = (Dim(H))\, Tr(S^{2}|_{x_H} H)$.

Theorem 1.12[6] Let H be a Hopf algebra over K. Then the following are equivalent:

All left $H-$comodules are completely reducible.

$<\lambda, 1> \neq 0$ for some $\lambda \in \int^{r}$.

$H = K1 \otimes C$ for some subcoalgebra C of H.

$<\lambda, 1> \neq 0$ for some $\lambda \in \int^{l}$

All right $H-$comodules are completely reducible.

Theorem 1.13[6] Let H be a cosemisimple Hopf algebra with antipode S. Then $S^{2}(C) = C$ for all simple subcoalgebras C of H.

2. The order of the antipode

Lemma 2.1 Suppose $\eta \in G(H^{*})$, $g \in G(H)$, $m, n \in H^{*}$ and $x \in L_{\eta}$ such that $m \to x = x \leftarrow n$. Then $m \in L_g$ and $n \in R_g$.

Proof Let $h^{*}, g^{*} \in H^{*}$. Then

$$
\begin{aligned}
(g^{*}h^{*}m)(x) &= \sum (g^{*}h^{*})(x_1 m(x_2)) \\
&= (g^{*}h^{*})(g) \\
&= g^{*}(g)h^{*}(g) \\
&= \sum g^{*}(m(x_2)x_1)h^{*}(g) \\
&= (g^{*}h^{*}(g)m)(x)
\end{aligned}
$$

which shows that $(g^{*}(h^{*}m - h^{*}(g)m))(x) = 0$, so $(h^{*}m - h^{*}(g)m)(x \leftarrow H^{*}) = 0$. But

$x \leftarrow H^{*} = H$, since $L_{\eta} \leftarrow \eta = L_{\varepsilon}$ and $L_{\varepsilon} \leftarrow H^{*} = H$ (applied for the dual of H^{φ}). This

shows that $h^{*}m = h^{*}(g)m$, and so $m \in L_g$. The fact that $n \in R_g$ is proved in a similar way.

Corollary 2.2 If $m \in H^{*}, x \in L_{\varepsilon}$, and $m \to x = 1$, then $m \in L_1$ and $x \leftarrow m = d$.

Proof If $h^{*} \in H^{*}$, then

$$
\begin{aligned}
h^{*}(x \leftarrow m) &= \sum h^{*}(x_2)m(x_1) \\
&= (hm^{*})(x) \\
&= h(d)m(x) \\
&\quad h^{*}(m(x)d)
\end{aligned}
$$

Applying ε to the relation $\sum m(x_2)x_1 = 1$ we get $m(x) = 1$. This shows that $x \leftarrow m = d$. The fact that $m \in L_1$ is proved by Lemma 2.1.

Lemma 2.3 Suppose $x \in L_{\eta}$, $g \in G(H)$, $m \in H^{*}$ such that $m \to x = g$. Then for any

$h^* \in H^*$ we have $\eta(g)h^*(1) = \sum h^*(x_1)m(gx_2)$.

Proof From the fact that $\Delta(h^*) = \sum h_1^* \otimes h_2^*$, $g = m \to x$ and $\eta(g)x = gx$, we have

$$\eta(g)h^*(1) = \sum \eta(g)h_1^*(g^{-1})h_2^*(g)$$
$$= h_1^*(g^{-1})h_2^*(m(x_2)x_1\eta(g))$$
$$= h_1^*(g^{-1})h_2^*(m(gx_2)gx_1)$$
$$= \sum h^*(g^{-1}gx_1)m(gx_2)$$
$$= \sum h^*(x_1)m(gx_2)$$

Lemma 2.4 Let $x \in L_\eta$, $g \in G(H)$, $m \in H^*$, and $\eta \in G(H^*)$ such that $m \to x = g$. Then for any $h \in H$ we have $S(g^{-1}(\eta \to h)) = (m \leftarrow h) \to x$.

Proof If $h^* \in H^*$. Then

$$h^*(S(g^{-1}(\eta \to h))) = \sum h^*(S(h_1)g)\eta(h_2)$$
$$= \sum \eta(g)h^*(S(h_1)g)\eta(g^{-1}h_2)$$
$$= \eta(g)((h^*S)\eta)(g^{-1}h)$$
$$= \sum ((h_1^*S)\eta)(g^{-1}h)\eta(g)h_2^*(1)$$
$$= \sum ((h_1^*S)\eta)(g^{-1}h)h_2^*(m(gx_2)x_1)$$
$$= \sum (h_1^*S)(g^{-1}h_1)\eta(g^{-1}h_2)h_2^*(m(gx_2)x_1)$$
$$= \sum h_1^*(S(h_1)g)h_2^*(m(g^{-1}h_3x_2)g^{-1}h_2x_1)$$
$$= \sum h_1^*(S(h_1)g^{-1}h_2x_1m(h_3x_2)$$
$$= \sum h^*(x_1m(hx_2)$$
$$= h^*((m \leftarrow h) \to x).$$

Remark 2.5 If we write the formula from Lemma2.4 for the Hopf algebras $H, H^{cop}, H^{op,cop}$ and H^{op}, we get that for any $h \in H$ the following relations hold:

Suppose $x \in L_\eta, m \to x = g$, then $S(g^{-1}(\eta \to h)) = (m \leftarrow h) \to x$;

Suppose $x \in R_\eta, m \to x = g$, then $S^{-1}((\eta \to h)g^{-1}) = (h \to m) \to x$;

Suppose $x \in R_\eta, x \leftarrow n = g$, then $S((h \leftarrow \eta)g^{-1}) = x \leftarrow (h \to n)$

Suppose $x \in L_\eta, x \leftarrow n = g$, then $S^{-1}(g^{-1}(h \leftarrow \eta)) = x \leftarrow (n \leftarrow h)$

In particular

If $x \in L_\varepsilon, m \to x = 1$, then $S(h) = (m \leftarrow h) \to x$ (2.1)

If $x \in R_\alpha = L_\varepsilon, m \to x = 1$, then $S^{-1}(\alpha \to h) = (h \to m) \to x$ (2.2)

If $x \in R_\alpha = L_\varepsilon, x \leftarrow n = d$, then $S((h \leftarrow \alpha)d^{-1}) = x \leftarrow (h \to n)$ (2.3)

If $x \in L_\varepsilon, x \leftarrow n = g$, then $S^{-1}(g^{-1}h) = x \leftarrow (n \leftarrow h)$ (2.4)

Theorem 2.6 For any $h \in H$ we have $S^4(h) = d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d$.

Proof Suppose $x \in L_\varepsilon = R_\alpha$, and $m \in H^*$ with $m \to x = 1$. Corollary 2.2 shows that $m \in L_1$ and $m \in L_1$ and $x \leftarrow m = d$. Moreover, we have

$$(S^4(h) \to m) \to x = S^{-1}(\alpha \to S^4(h))$$

(by (2.2))

$$= S^{-1}(S^4(\alpha \to h))$$
$$= S(S^2(\alpha \to h))$$
$$= (m \leftarrow S^2(\alpha \to h)) \to x$$

(by (2.1))

Since the map from H^* to H, sending $h^* \in H^*$ to $h^* \to x \in H$ is bijective, we obtain

$$S^4(h) \to m = m \leftarrow S^2(\alpha \to h).$$

On the other hand,

$$x \leftarrow (m \leftarrow S^2(\alpha \to h)) = S^{-1}(d^{-1}S^2(\alpha \to h))$$

(by (2.4))

$$= S^{-1}(S^2(d^{-1}(\alpha \to h)))$$
$$= S(d^{-1}(\alpha \to h))$$
$$= S(d^{-1}(\alpha \to h)d\ d^{-1})$$
$$= S(((d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \leftarrow \alpha)d^{-1})$$
$$= x \leftarrow ((d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \to m)$$

(by (2.3))

Since the map $h^* \mapsto (x \leftarrow h^*)$ from H^* to H is bijective, we obtain that

$$m \leftarrow S^2(\alpha \to h) = (d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \to m$$

We got that

$$S^4(h) \to m = (d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \to m$$

then the formula follows from the bijectivity of the map $h \mapsto (h \to m)$ from H to H^*.

Theorem 2.7 Let H be a finite dimensional Hopf algebra. Then the antipode S has finite order.

Proof By Theorem 2.6, we obtain by induction that

$$S^{4n}(h) = d^{-n}(\alpha^n \to h \leftarrow \alpha^{-n})d^n$$

for any positive integer n. Since $G(H)$ and $G(H^*)$ are finite groups, their elements have finite orders, so there exists p for which $d^p = 1$ and $\alpha^p = \varepsilon$. Then it follows that $S^{4p} = I$.

3 Characterizations of semisimple Hopf algebras

Semisimlpe Hopf algebras are finite-dimsnsional by part (2) of Lemma 1.3. We characterize finite-dimensional Hopf algebras which are semisimple in the algebraically closed characteristic zero case. To this end we calculate a trace.

Lemma3.1 If C is a simple coalgebra over K, and T is a diagonalizable coalgebra automorphism of C. The

$$Tr(T) = (\sum_{i=1}^n \lambda_i)(\sum_{i=1}^n \lambda_i^{-1})$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues for T.

Proof By lemma1.5 we obtain that $C \cong C_n(K)$ for some $n \geq 1$. Thus we may assume

$C = C_n(K)$. The crux of the proof will be to show that there is a simple left coideal M of C

such that $T(M) \subseteq M$. Necesarily $Dim(M) = n$.

T^* is an algebra automorphism of $C^* = M_n(K)$ By Skolem-Noether Theorem, there is an invertible matrix $u \in M_n(K)$ such that $T^*(a) = uau^{-1}$ for all $a \in M_n(K)$ Identify $C^* = M_n(K)$ with $End(V)$,where V is an $n-$ dimensional vector space over K. Since K is algebraically closed, u has an eigenvalue $\lambda \in K$. Let $v \in V$ be a non-zero vector satisfying $u(V) = \lambda v$. Regard $End(V)$ and V as left $End(V)-$ modules via function composition and evaluation respectively. Then V is a simple module and the evaluation map

$e_v : End(V) \to V$ given by $e_v(a) = a(v)$ for all $a \in End(V)$

is a module map. Therefore $L = Ker(e_v) = \{a \in End(V) | a(v) = 0\}$ is a maximal left ideal of

$End(V)$ of codimension $n^2 - n$. Observe that $T^*(L) \subseteq L$. Set $M = L^{\perp}$.Then M is a minimal left coideal of C by Lemma 1.4 and $T(M) \subseteq M$ by Lemma 1.6. and Using Remark1.7 we see that $Dim(M) = n$.

Since T is diagonalizable and $T(M) \subseteq M$ it follows that the restriction $T|M$ is diagonalizable. Let $\{m_1, m_2, \cdots, m_n\}$ be a basis of eigenvectors for $T|M$ and let $\lambda_1, \cdots, \lambda_n \in K$

satisfy $T(m_i) = \lambda_i m_i$ for all $1 \le i \le n$. Then $\lambda_1, \cdots, \lambda_n$ are non-zer scalars since $T|M$ is noe-

one. For each $1 \le i \le n$ write $\Delta(m_i) = \sum_{j=1}^{n} c_{i,j} \otimes m_j$. Then the $c_{i,j}$'s satisfy the comatrix identities and thus span a non-zero subcoalgebra D of C. Since C is simple $D = C$. Since

$Dim(C) = n^2$ necessarily the $c_{i,j}$'s from a basis for C. Applying $T \otimes T$ to both sides of the

equation for $\Delta(m_i)$ yields $\sum_{j=1}^{n} \lambda_i c_{i,j} \otimes m_j = \sum_{j=1}^{n} T(c_{i,j}) \otimes \lambda_j m_j$. Therefore $T(c_{i,j}) =$

$\lambda_i \lambda_j^{-1} c_{i,j}$ for all $1 \le i, j \le n$. Since $\{c_{i,j}\}_{1 \le i,j \le n}$ is a basis for C we calculat

$$\mathrm{Tr}(T) = \sum_{i,j=1}^{n} \lambda_i \lambda_j^{-1} = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i^{-1})$$

Theorem 3.2 Let H be a Hopf algebra with antipode S over K. Then the following are equivalent.

H is cosemisimple.

$\mathrm{Tr}(S^2) \neq 0$.

H is semisimple.

$S^2 = 1_H$.

$\omega : H \to K$ defined by $\omega(a) = \mathrm{Tr}(r(a)$ for all $a \in H$ is a right integral for H.

Proof (1)$\Rightarrow$(2). Since H is cosemisimple it is the direct sum of its simple subcoalgebras . Let C be a simple subcoalgebra of H. Then $S(C) = C$ By Theorem1.13 . Now S^2 has finite order by part (1) of Theorem Proposition1.11. Since K is algebraically closed of

characteristic zero S^2 is diagonalizable. Thus $\mathrm{Tr}(S^2) = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i^{-1})$ where $\lambda_1, \cdots, \lambda_n$ are roots of unity by Lemma 3.1. Since the characteristic of K is zero we may assume that $\lambda_1, \cdots, \lambda_n \in$ C, the field of compex mubers. Thus

$$\mathrm{Tr}(S^2|C) = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i^{-1}) = (\sum_{i=1}^{n} \lambda_i)(\overline{\sum_{i=1}^{n} \lambda_i}) = \left|\sum_{i=1}^{n} \lambda_i\right|^2$$

is a non-negative real number. Therefore $\mathrm{Tr}(S^2) = 1 + \sum_C \mathrm{Tr}(S^2|C) \ge 1$, where C runs over the simple subcoalgebras $C \neq K1$ of H. We have shown that $\mathrm{Tr}(S^2) \neq 0$.

(2)$\Rightarrow$(3). It is pretty obvious by Proposition1.10.

(3)$\Rightarrow$(4). Assume that H is semisimple. Then H^* is cosemisimple. We have just show H^*

is semisimple; thus H is semisimple and cosemisimple. In particular $\mathrm{Tr}(S^2) \neq 0$. Now $\mathrm{Tr}(S^2) = (Dim(H)) \mathrm{Tr}(S^2|_{x_H} H)$ by part (3) of Proposition 1.11 and $S^4 = 1_H$ by part (2) of Porposition1.11. Since the characteristic of K is not 2, the last equation implies S^2 is a diagonalizable endomorphism of H with eigenvalues ± 1. Choose a basis of eigenvectors for S^2. Let n_+ be the number of basis vectors belonging to the eigenvalue 1 and let n_- be the number belonging to -1. By the preceding trace formula $n_+ - n_- = (n_+ + n_-)m$ for some integer m which is not zero since $\mathrm{Tr}(S^2) \neq 0$.Squaring both sides of this equation yields

$$-2n_+n_- = (m^2 - 1)n_+^2 + 2m^2 n_+ n_- + (m^2 - 1)n_-^2 \ge 0.$$

Therefore $n_+ n_- = 0$. Since $n_+ \neq 0$ necessarily $n_- = 0$. We have shown $S^2 = 1_H$.

(4)$\Rightarrow$(5). That it is very simple follows by part (2) of Proposition1.9.

(5)$\Rightarrow$(1). Since $\omega(1) = Dim(H)1 \neq 0$, thus our proof is complete by Theorem1.12.

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