DOI: 10.18686/ahe.v6i17.5481

The Boundedness of Maximal Operators in Generalized Orlicz-Campanato Spaces

Lijun Bao[1] **, Yanlong Shi**[2]

[1]Ningbo Open University, Ningbo Zhejiang,315100, P.R. China

[2]Zhejiang Pharmaceutical University, Ningbo, Zhejiang, 315100, P.R. China

Abstract: In this paper, the boundedness for the Hardy-Littlewood maximal operators in generalized Orlicz-Campanato space $L^{\Phi,\phi}$ will be proved.

Keywords: Hardy-Littlewood maximal operators； N-function；Generalized Orlicz-Morrey space.

1. Introduction

The Hardy-Littlewood maximal operator Mf is defined by

$$Mf(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|dy \qquad f \in L_{loc}(R^n),$$

where the supremum is taken over all cubes Q containing x and |Q| denotes the Lebesgue measure of Q. A cube in R^n will always mean a compact cubic interval with sides parallel to the axes.

In 1981, Bennett, Devoe, and Sharply[1] study the boundedness of maximal operator in BMO spaces and proved that:

Theorem A[1] Let $f \in BMO$, if inf $Mf(x) < \infty$, then $Mf(x) \in BMO$,and

$$\| Mf(x) \|_{BMO} \leq \| f \|_{BMO}.$$

In 1989, Chen Jiecheng showed a new and simpler proof of Theorem A (see[2]). As an extension of the BMO spaces, it is natural and important to study the boundedness of maximal operator in Campanato spaces, one can see [3] for the recent progress. In this paper, we will establish the boundedness in generalized Orlicz-Campanato space $L^{\Phi,\phi}$ for Hardy-Littlewood Maximal Operators.

In this paper, the N-function Φ(s) involved is given by

$$\Phi(s) = \int_0^s \varphi(t)dt \qquad s \geq 0,$$

where $\varphi(t)$ is a positive nondecreasing function defined for $t > 0$ and $\varphi(0) = 0$. An N-function Φ is a convex function and is said to satisfy the Δ_2 condition in $(0,\infty)$ if

$$\Phi(2s) \leq C\Phi(s) \qquad (1.1)$$

for all s > 0.

Definition 1 Let Φ be as above, we define the module of generalized Orlicz- Campanato space

$$L^{\Phi,\phi} = \{ f \in L^1_{loc}(R^n): \sup_{y\in R^n, r>0} \frac{1}{\phi(r)}\int_{B(y,r)} \Phi(| f(x) - f_B | dx < \infty)\}$$

as following:

$$\| f \|_{L^{\Phi,\phi}} = \sup_{y\in R^n, r>0} \frac{1}{\phi(r)}\int_{B(y,r)} \Phi(| f(x) - f_B |)dx.$$

where $B(y,r)$ denotes a ball with center y and radius r , $f_B =| B(y,r)|^{-1} \int_{B(y,r)} f(x)\, dx$. $\phi(r)$ is a nondecreasing function in $(0,\infty)$ and satisfies following doubling condition, there exist a constant $1 < D < 2^{n+\alpha}$ with $\alpha > 0$ such that

$$\phi(2r) \le D\phi(r) \tag{1.2}$$

for all $r > 0$

Definition 2 The lower index $q_\Phi = \lim_{\lambda\to 0^+} \frac{\log h(\lambda)}{\log \lambda}$ and upper index $p_\Phi = \lim_{\lambda\to +\infty} \frac{\log h(\lambda)}{\log \lambda}$,

where $h(\lambda) = \sup_{t>0}[\Phi(\lambda t) / \Phi(t)]$

It is easily known that (see[4][5])

$$1 < q_\Phi \le p_\Phi < \infty \tag{1.3}$$

Now we state our main result.

Theorem 1 Let N-function $\Phi(u)$ satisfies the Δ_2 condition, and ϕ satisfies the doubling condition (1.2), if $\inf_{x\in R^n} Mf < \infty$, then $Mf(x) \in L^{\Phi,\phi}$, and $\| Mf \|_{L^{\Phi,\phi}} \le C \| f \|_{L^{\Phi,\phi}}$.

We remark that if Φ(s) = |s|^p, $\phi(r) = r^n$, then we can get the boundedness of Hardy-Little maximal operator in BMO(P) spaces which is equivalent with BMO(see[6]), and if Φ(s) = |s|^p, $\phi(r) = r^\beta$, $0 < \beta < n+\alpha$ with $0 < \alpha < p-1$, then we can get the boundedness of Hardy-Little maximal operator in classical Campanato spaces, and throughout this paper, the letter C will denote the absolute positive constant, which may have different value in each line.

2. Lemmas and the proof of Theorem 1

First, we propose the following properties of Φ.

Lemma 1 If Φ(s) satisfies the Δ_2 condition then Φ(s) essentially equals to $s\varphi(s)$.

Proof In fact, $\Phi(s) = \int_0^s \varphi(t)dt \le \int_0^s \varphi(s)dt \le s\varphi(s)$, also, $\Phi(s) = \int_0^s \varphi(t)dt \ge$

$$\int_{s/2}^s \varphi(t)dt \ge \int_{s/2}^s \varphi(\frac{s}{2})dt = \frac{s}{2}\varphi(\frac{s}{2}) \ge Cs\varphi(s)\cdot$$

Lemma 2 If Φ(s) satisfies the Δ_2 condition, p_Φ, q_Φ are the indexes defined above, then we get

$$\Phi(\lambda t) \le C_1\lambda^p\Phi(t), \forall t \ge 0, 0 \le \lambda \le 1, 0 < p < q_\Phi \tag{2.1}$$

$$\Phi(\lambda t) \le C_2\lambda^q\Phi(t), \forall t \ge 0, \lambda > 1, p_\Phi < q < \infty \tag{2.2}$$

Proof Let $0 < p < q_\Phi$, from the definition of q_Φ , there exists a small positive number λ_0 such that $\frac{\log h(\lambda)}{\log \lambda} > p$ for $0 < \lambda \le \lambda_0$, so we get $\Phi(\lambda t) \le \lambda^p\Phi(t)$ for $0 < \lambda \le \lambda_0$ and any $t > 0$. On the other hand , for $\lambda_0 \le \lambda < 1$ there exists a proper integer k such that

$2^{k-1}\lambda_0 \le \lambda < 2^k\lambda_0 < 2$, then we deduce

$$\Phi(\lambda t) \le \Phi(2^k\lambda_0 t) < \lambda_0{}^p\Phi(2^k t)$$

$$\le \lambda_0{}^p C^k\Phi(t) \le 2^p(\frac{C}{2^p})^k\lambda^p\Phi(t) \le C\lambda^p\Phi(t)$$

where we have used the fact $1 \le k < \log_2 \frac{2}{\lambda_0}$.

Similarly, we have $\Phi(\lambda t) \le C_2 \lambda^q \Phi(t), \forall t \ge 0, \lambda > 1, p_\Phi < q < \infty$. This completes the proof.

Lemma 3 If Φ(s) satisfies the Δ2 condition, then

$$\int_0^u \frac{\Phi(s)}{s^2} ds \le C \frac{\Phi(u)}{u} \tag{2.3}$$

Proof Since $q_\Phi > 1$, so we can take $1 < m < q_\Phi$, by Lemma 2, we have $\Phi(su) \le Cs^m \Phi(u)$ for all $0 \le s \le 1$, so

$$\int_0^u \frac{\Phi(s)}{s^2} ds = \int_0^1 \frac{\Phi(us)}{u^2 s^2} u ds = \frac{1}{u} \int_0^1 \frac{\Phi(us)}{s^2} ds \le \frac{C}{u} \int_0^1 \frac{s^m}{s^2} \Phi(u) ds$$
$$= C \frac{\Phi(u)}{u} \int_0^1 \frac{1}{s^{2-m}} ds = C \frac{\Phi(u)}{u}$$

Lemma 4 Let $1 < p < q_\Phi$ and $0 < \alpha < p-1$. Suppose ϕ satisfies the doubling condition (1.2) with $1 < D < 2^{n+\alpha}$, and Φ is as above. If $f \in L^{\Phi,\phi}$, then

$$\int_{R^n} \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy \le C\phi(r_0) {r_0}^{-n} \| f \|_{L^{\Phi,\phi}}$$

Proof Let $B = B(x_0, r_0)$, $B(k) = B(x_0 2^k r_0)$

$$\int_{R^n} \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy$$
$$\le \sum_{k=1}^{\infty} \int_{B(k)-B(k-1)} \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy$$
$$+ \int_B \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy$$
$$= I_1 + I_2$$

Obviously,

$$I_2 = \int_B \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy$$
$$\le C{r_0}^{-n} \frac{\phi(r_0)}{\phi(r_0)} \int_B \Phi(|f - f_B|) dy$$
$$\le C{r_0}^{-n} \phi(r_0) \| f \|_{L^{\Phi,\phi}}$$

Now, we need to estimate I_1. Since Φ is a convex function and satisfies the Δ2 condition, we have

$$I_1 = \sum_{k=1}^{\infty} \int_{B(k)-B(k-1)} \frac{{r_0}^{p-1}}{|r_0 + |y - x_0||^{n+p-1}} \Phi(|f - f_B|) dy$$
$$\le \sum_{k=1}^{\infty} \frac{C{r_0}^{p-1}}{(2^k r_0)^{n+p-1}} \int_{B(k)} \Phi(|f - f_B|) dy$$
$$\le \sum_{k=1}^{\infty} \frac{C{r_0}^{p-1}}{(2^k r_0)^{n+p-1}} \left(\int_{B(k)} \Phi(|f - f_{B(k)}|) dy + \int_{B(k)} \Phi(|f_{B(k)} - f_B| dy)\right)$$
$$= J_1 + J_2$$

Recall $\phi(r)$ satisfies the doubling condition (1.2) with $1 < D < 2^{n+\alpha}$ and $0 < \alpha < p-1$, it is easy to see that

$$\begin{aligned} J_1 &= C\sum_{k=1}^{\infty}\frac{r_0^{\,p-1}}{(2^k r_0)^{n+p-1}}\int_{B(k)}\Phi(|f-f_{B(k)}|)dy \\ &\le C\sum_{k=1}^{\infty}\frac{r_0^{\,p-1}\phi(2^k r_0)}{(2^k r_0)^{n+p-1}}\frac{1}{\phi(2^k r_0)}\int_{B(k)}\Phi(|f-f_{B(k)}|)dy \\ &\le C\sum_{k=1}^{\infty}\frac{r_0^{\,p-1}D^k}{(2^k r_0)^{n+p-1}}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \\ &\le C\sum_{k=1}^{\infty}\frac{r_0^{\,p-1}}{(2^k r_0)^{n+p-1}}(2^{n+\alpha})^k\phi(r_0)\|f\|_{L^{\Phi,\phi}} \\ &\le C\sum_{k=1}^{\infty}r_0^{-n}(2^k)^{\alpha-p+1}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \\ &\le Cr_0^{-n}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \end{aligned}$$

Finally, we need to estimate J_2. Using Jensen inequally and (1.2), we obtain

$$\begin{aligned} \Phi(|f_{B(k+1)}-f_{B(k)}|) &= \Phi(|\frac{1}{|B(k)|}\int_{B(k)}(f_{B(k+1)}-f)dx|) \\ &\le \frac{1}{|B(k)|}\int_{B(k)}\Phi(|f-f_{B(k+1)}|)dx \\ &\le \frac{\phi(2^{k+1}r_0)}{|B(k)|\phi(2^{k+1}r_0)}\int_{B(k+1)}\Phi(|f-f_{B(k+1)}|)dx \\ &\le C\frac{D^{k+1}\phi(r_0)}{(2^k r_0)^n}\|f\|_{L^{\Phi,\phi}} \end{aligned} \tag{2.4}$$

Therefore, by (2.2), (2.4) and the convex property of Φ,

$$\begin{aligned} \Phi(|f_{B(k)}-f_B|) &\le \Phi(k\sum_{i=0}^{k-1}\frac{|f_{B(i+1)}-f_{B(i)}|}{k}) \\ &\le k^{q-1}\sum_{i=0}^{k-1}\Phi(|f_{B(i+1)}-f_{B(i)}|) \\ &\le Ck^{q-1}\sum_{i=0}^{k-1}\frac{D^{i+1}\phi(r_0)}{(2^i r_0)^n}\|f\|_{L^{\Phi,\phi}} \\ &\le Ck^{q-1}\sum_{i=0}^{k-1}\frac{2^{(n+\alpha)(i+1)}}{(2^i r_0)^n}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \\ &\le Ck^{q-1}\sum_{i=0}^{k-1}(2^\alpha)^i r_0^{-n}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \\ &\le Ck^{q-1}r_0^{-n}\phi(r_0)\|f\|_{L^{\Phi,\phi}}\frac{1-(2^\alpha)^k}{1-2^\alpha} \\ &\le Ck^{q-1}(2^\alpha)^k r_0^{-n}\phi(r_0)\|f\|_{L^{\Phi,\phi}} \end{aligned} \tag{2.5}$$

From (2.5) and assumption $0 < \alpha < p-1$, we have

$$J_2 = C\sum_{k=1}^{\infty}\frac{r_0^{\ p-1}}{(2^k r_0)^{n+p-1}}\int_{B(k)}\Phi(|f_{B(k)}-f_B|)dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{r_0^{\ p-1}}{(2^k r_0)^{n+p-1}}|B(k)|k^{q-1}(2^{\alpha})^k r_0^{\ -n}\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{r_0^{\ p-1}}{(2^k r_0)^{n+p-1}}k^{q-1}(2^{n+\alpha})^k\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

$$\leq C\sum_{k=1}^{\infty}k^{q-1}(2^k)^{\alpha-p+1}r_0^{\ -n}\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

$$\leq Cr_0^{-n}\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

Combining the above estimate of J_1 and J_2, obviously,

$$I_1 \leq J_1 + J_2 \leq Cr_0^{\ -n}\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

The proof of the Lemma 4 is complete.

Lemma 5 Let $\varphi(x)=\chi_{B(0,1)}(x)$, $\varphi_t(x)=t^{-n}\varphi(x/t)$, $B=B(x_0,r_0)$, $\tilde{B}=B(x_0,4r_0)$. Suppose ϕ satisfies the doubling condition (1.2) with $1<D<2^{n+\alpha}$ and $0<\alpha<p-1$, where $1<p<q_{\Phi}$, and that Φ is N-function satisfying Δ_2 condition.

Then

$$\Phi(\sup_{t>0,x\in B}|\int_{\tilde{B}^c}(\varphi_t(x-y)-\varphi_t(x_0-y))(f(y)-f_{\tilde{B}})dy|)\leq C\phi(r)r_0^{\ -n}\|f\|_{L^{\Phi,\phi}}$$

for any $f\in L^{\Phi,\phi}$.

Proof Put $B_{x,x_0}(t)=B(x_0,t)\Delta B(x,t)$ (the symmetric difference), if $t<3r_0$ and $x\in B$, then $B_{x,x_0}(t)\cap\tilde{B}^c=\varnothing$ and $|\varphi_t(x-y)-\varphi_t(x_0-y)|=t^{-n}\chi_{B(0,1)}(t)(y)$, so

$$\Phi(\sup_{t>0,x\in B}|\int_{\tilde{B}^c}(\varphi_t(x-y)-\varphi_t(x_0-y))(f(y)-f_{\tilde{B}}(y)dy|)$$

$$\leq\Phi(\sup_{t>3r_0,x\in B}|\int_{\tilde{B}^c}(\varphi_t(x-y)-\varphi_t(x_0-y))(f(y)-f_{\tilde{B}}(y)dy|)$$

$$\leq C\sup_{t>3r_0,x\in B}\Phi(|\int_{t-r_0<|y-x_0|<t+r_0}t^{-n}|f(y)-f_{\tilde{B}}|dy|)$$

$$\leq C\sup_{t>3r_0,x\in B}\Phi(\frac{t^{n-1}r_0}{t^n}\frac{1}{t^{n-1}r_0}\int_{t-r_0<|y-x_0|<t+r_0}|f(y)-f_{\tilde{B}}|dy)$$

$$\leq C\sup_{t>3r_0,x\in B}(\frac{r_0}{t})^p\Phi(\frac{1}{t^{n-1}r_0}\int_{t-r_0<|y-x_0|<t+r_0}|f(y)-f_{\tilde{B}}|dy)$$

$$\leq C\sup_{t>3r_0,x\in B}(\frac{r_0}{t})^p\frac{1}{t^{n-1}r_0}\int_{t-r_0<|y-x_0|<t+r_0}\Phi|f(y)-f_{\tilde{B}}|dy)$$

$$\leq C\int_{R^n}\frac{r_0^{\ p-1}}{|r_0+|y-x_0||^{n+p-1}}\Phi|f(y)-f_B|dy)$$

$$\leq C\phi(r_0)r_0^{\ -n}\|f\|_{L^{\Phi,\phi}}$$

where we have found that $(t+r_0)-(t-r_0)\sim C_n t^{n-1}r_0$, and have used (2.1) in the fourth inequality, Jensen inequality in the fifth and Lemma 4 in the last.

Now, we return to the proof of Theorem 1.

Proof Assume that $f\in L^{\Phi,\phi}$ and $f>0$, fix a ball $B=B(x_0,r_0)$, $\tilde{B}=B(x_0,4r_0)$, write f as

$$f = f_{\tilde{B}} + (f - f_{\tilde{B}})\chi_{\tilde{B}} + (f - f_{\tilde{B}})\chi_{\tilde{B}^c} = \sum_{i=1}^{3} f_i$$,

Note that Φ is a N-function and $M(f_{\tilde{B}})(x) = f_{\tilde{B}}$, we have

$$\int_B \Phi(|Mf - f_{\tilde{B}} - Mf_3(x_0)|)dx$$
$$\leq \int_B \Phi(|Mf_{\tilde{B}} + Mf_2(x) + Mf_3(x) - Mf_3(x_0) - f_{\tilde{B}}|)dx$$
$$\leq \int_B \Phi(|Mf_2(x) + Mf_3(x) - Mf_3(x_0)|)dx$$
$$\leq \int_B \Phi(Mf_2(x))dx + \int_B \Phi(|Mf_3(x) - Mf_3(x_0)|)dx$$
$$= K_1 + K_2$$

Since $\phi(r)$ satisfy the doubling condition, we obtain

$$K_1 \leq \int_0^\infty |\{x \in R^n : Mf_2(x) > s\}| \varphi(s)ds$$
$$\leq C\int_0^\infty [\frac{1}{s}\int_{f_2(x)>s} |f_2(x)|dx]\frac{\Phi(s)}{s}ds$$
$$\leq C\int_{R^n} [f_2(x)\int_0^{|f_2(x)|}\frac{\Phi(s)}{s^2}ds]dx$$
$$\leq C\int_{R^n} \Phi(|f_2(x)|)dx$$
$$\leq C\int_{\tilde{B}} \Phi(|f - f_{\tilde{B}}|)dx$$
$$\leq C\phi(r_0)\|f\|_{L^{\Phi,\phi}}$$

Where we have used Lemma 1 in the second inequality and Lemma 3 in the fourth .

Now, we need to estimate K_2. Note that

$$|Mf_3(x) - Mf_3(x_0)| \leq C|\sup_{x\in B, t>0} \varphi_t * f_3(x) - \sup_{x\in B, t>0} \varphi_t * f_3(x_0)|$$
$$\leq C\sup_{x\in B, t>0} |\varphi_t * f_3(x) - \varphi_t * f_3(x_0)|$$

Hence, by Lemma 5, we have

$$\Phi(|Mf_3(x) - Mf_3(x_0)|) \leq \Phi(C\sup_{x\in B, t>0} |\varphi_t * f_3(x) - \varphi_t * f_3(x_0)|)$$
$$\leq C\Phi(\sup_{x\in B, t>0} |\int_{\tilde{B}^c} (\varphi_t(x-y) - \varphi_t(x_0 - y))(f(y) - f_{\tilde{B}})dy|)$$
$$\leq C\phi(r_0)r_0^{-n}\|f\|_{L^{\Phi,\phi}}$$

Suppose that there is an x_0 so that $Mf(x_0) < \infty$, then $Mf_3(x_0) < f_{\tilde{B}} + Mf(x_0) < \infty$, combining the above estimate, we have

$$\Phi(Mf_3)(x) \leq C\phi(r_0)r_0^{-n}\|f\|_{L^{\Phi,\phi}} + \Phi(Mf_3(x_0)) < \infty,$$

So, as $Mf_3(x_0) < \infty$, we have

$$\int_B \Phi(|Mf - f_{\tilde{B}} - Mf_3(x_0)|)dx \leq C\phi(r_0)r_0^{-n}\|f\|_{L^{\Phi,\phi}}$$

Thus

$$\| Mf \|_{L^{\Phi,\phi}} \leq C \| f \|_{L^{\Phi,\phi}}$$

The proof of the Theorem 1 is complete.

References:

[1]Bennett, C, Devor. R. A. Sharpley. R. Weak- L^{∞} and BMO[J], Ann Math, 1981,113: 601-611.

[2]Chen Jiecheng. Some remarks about Hardy-Littlewood maxial operator[J]. Journal of Hangzhou university, 1989, 116(3): 259-265.

[3]X-X. Tao, Q-Q. Chen, The boundedness of maximal function in Orlicz-Campanato spaces of homogeneous type. Georgian Mathematical Journal, 2008, 15(2): 377-388.

[4]KokilashviliV, KrbecM. Weighted Inequalities in Lorentz and Orlicz Spaces. Singa- pore: World Scientific, 1991, 171-209.

[5]Kerman R, Torchinsky A. Integral inequalities with weights for the Hardymaximal function. Studia Math, 1982, 71: 277-284.

[6]Tao Xiangxing. Weighted Φ-modulo Inequalities for the Hardy-Littlewood Maxial Operators. Northteastern Math. J. 9(4)(1993), 469-476.