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Matching Outcomes in SOM vs. Boston Mechanism

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Abstract: We critically examine the Sequential Offer Mechanism (SOM) in the context of school choice, comparing it to the Boston Mechanism. Our focus is on the properties of stability within SOM. When students report their true preferences, under the condition that every school fills its capacity, the matching outcome of the Sequential Offer Mechanism (SOM) is identical to that of the Boston Mechanism.

Keywords: School Choice; Sequential Offer Mechanism (SOM)

1. Introduction

The task of assigning students to educational institutions, whether it's K-12 schools, undergraduate colleges, or advanced degree programs, holds profound implications for individual trajectories as well as societal prosperity. The Deferred Acceptance algorithm (DA), originally formulated by Gale and Shapley, is frequently celebrated for its equitable and incentivizing features, often gaining preference over Gale's Top Trading Cycles (TTC) in academic and policy discussions.

However, the educational landscape in China, particularly in the realms of master's and doctoral program admissions, provides an interesting deviation from conventional, synchronized systems. In these cases, offers are made according to staggered timelines dictated by each institution's specific calendar or set of priorities. This asynchrony adds a layer of complexity to the already challenging problem of educational placement, as students must strategize around not just program rankings but also offer timings. There is growing evidence to suggest that certain sequential allocation mechanisms could prove more efficient than their simultaneous counterparts in such settings.

Certainly, it's crucial to critically assess the effectiveness and limitations of different school admission algorithms, such as the Boston Mechanism and the Sequential Offer Mechanism (SOM). Each algorithm comes with its own set of advantages and drawbacks that deeply influence students' behaviors and outcomes.

The Boston Mechanism is often lauded for its simplicity and ease of implementation. However, this mechanism promotes strategic manipulation by students when it comes to ranking their preferred schools. Specifically, if a student doesn't gain admission to their top choice, their ranking at their second-choice school suffers in comparison to those who prioritized that school. In this way, the Boston Mechanism indirectly penalizes students for revealing their actual preferences, inducing them to employ game-theoretic strategies in their school rankings.

To summarize, while the Boston Mechanism's limitations revolve around encouraging strategic behavior due to its priority treatment, SOM adds an additional layer of complexity through its timing constraints. Both algorithms, therefore, compel students to make carefully calculated decisions rather than simply listing their true preferences, thereby affecting the overall fairness and effectiveness of the school allocation process.

Example 1. Consider three schools: School A, School B, and School C, each with one seat available. These schools announce their admissions at different times, with School A being the earliest and School C being the latest. Now, let's consider a student, Emma, who prefers School C the most, followed by School B, and finally School A.

Scenario 1: Standard Boston Mechanism

Emma ranks School C as her first choice and School B as her second choice.

If she is not admitted to School C, her priority at School B would be lower than those who ranked School B as their first choice.

Scenario 2: Sequential Offer Mechanism (SOM)

Emma again ranks School C as her first choice and School B as her second choice.

School A announces admissions first. Since Emma ranked it last, she misses the opportunity to go there.

Next, School B announces admissions. Assuming Emma also did not get into School C, her priority here is not just lowered; she may have completely missed the chance as the admission period for School A has already passed.

By the time School C announces admissions, Emma finds out she did not get in. Now, she has no options left.

2. 1 The Sequential Offer Mechanism (SOM)

Step 1: Initialization

Mark all students as "unmatched."; Set the remaining seats for all schools to their initial capacities, i.e., q_{si}.

Step 2: Sort Schools by Admission Order

Sort the schools based on their admission order .

Step 3: Sequential Offers

Iterate through each school s in accordance with the admission order o and perform the following steps:

Collect Applications: Gather applications from all students who consider s_i as their current best option (i.e., it is ranked highest among their unmatched options).

Rank and Admit: Utilize the school s_i's priority list f_{si} to rank the applying students and admit the top q_{si} students based on this ranking.

Update Status: Mark admitted students as "matched" and remove all schools from their preference list that are ranked lower than s_i.

Update Capacities: Update the remaining seats for school s_i.

Note: Once the admission time for school s_i has passed, the school can no longer admit any more students, even if it has not filled all its seats.

Step 4: Termination

Upon the completion of iterations for all schools, return the final matching μ.

To illustrate the operation of the Sequential Offer Mechanism (SOM), let us consider the following example.

Example 2. Sequential Offer Mechanism(SOM)

Scenario:

Set of students $I = \{i_1, i_2, i_3, i_4, i_5\}$

Set of schools $S = \{A, B, C\}$

School capacity vector q = (3,1,1)

Admission order $o = (A, C, B)$

School priorities and student preferences are as follows:

$f_A: i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5$
$f_B: i_5 \succ i_4 \succ i_3$
$f_C: i_2 \succ i_1 \succ i_5$
$P_{i_1}: A \succ B \succ C$
$P_{i_2}: A \succ C \succ B$
$P_{i_3}: B \succ A$
$P_{i_4}: B \succ C$
$P_{i_5}: C \succ A \succ B$

SOM Procedure

Step 1: Initialization

All students are marked as "unmatched."

Remaining seats: $A = 3, B = 1, C = 1$

Step 2: Sort Schools by Admission Order

Schools in order: A,C,B

Step 3: Sequential Offers

Round for School A:

Applicants: i_1, i_2

Admitted: i_1, i_2

Remaining seats for A: 0

Round for School C:

Applicants: i_5

Admitted: i_5

Remaining seats for C:0

Round for School B:

Applicants: i_3, i_4

Admitted: i_4 (since i_4 has higher priority at B)

Remaining seats for B: 0

Step 4: Termination

Final Matching: $i_1 \longrightarrow A, i_2 \longrightarrow A, i_3 \longrightarrow \text{Unmatched}, i_4 \longrightarrow B, i_5 \longrightarrow C$.

This example illustrates the process of matching students to schools under the SOM mechanism, especially when there is a sequential timing in admissions. Note that even though School A did not fill all its slots in the final round (only admitted), it could not admit any more students since its admission time had passed.

2.2 The Boston Mechanism

Step 1: Initialization Label all students as "unmatched." Initialize the remaining seat count for each school to its full capacity, denoted as q_{si}.

Step 2: Prioritize Preferences

For each student, identify the school that ranks highest on their preference list and is still available as an option.

Step 3: Admission Rounds

Iterate through multiple rounds of admissions, performing the following steps for each round:

Compile Applications: Assemble applications from students who have selected s_i as their top unmatched choice.

Rank and Admit: Use the school s_i' priority ordering f_{si} to sort the applicants and admit the highest-ranking q_{si} students.

Update Status: For students who are admitted, label them as "matched" and eliminate any schools from their list that are ranked below s_i.

Update Capacities: Update the remaining seat count for each school s_i.

Step 4: Review and Adjustment

After the first round, return to Step 2 and repeat the process for students who are still unmatched, but this time using reduced capacities for the schools that have already admitted students.

Step 5: Termination

Once all rounds are completed, or when all schools have reached capacity, finalize the student-to-school matching, denoted as μ.

Proof:

Let μ_{SOM}:I⟶S and I⟶S represent the matchings obtained through the SOM and the Boston Mechanism, respectively. We assume that $\forall s \in S, |\mu_{SOM}^{-1}(s)| = |\mu_{Boston}^{-1}(s)| = q_s$, where q_s is the capacity of school .Initially, for each student $i \in I, \mu_{SOM}(i) = \mu_{Boston}(i) = \text{Unmatched}$. Additionally, each school has available seats in both mechanisms.

We aim to prove that $\mu_{SOM}(i) = \mu_{Boston}(i), \forall i \in I$.

Consider any round in both mechanisms. Given the full-capacity assumption, the set of students A_s who apply to a school s must be the same in both mechanisms. This is because any student i will apply to his or her highest-ranked school s that has not yet reached its capacity in either mechanism. Furthermore, each school s admits students based on a common priority list f_s. Therefore, the subset of students $A_s' \subseteq A_s$ who get admitted to s in round t will be the same under both mechanisms. This argument applies to each round and each school, so we conclude that $\mu_{SOM}(i) = \mu_{Boston}(i)$ for all $i \in I$, proving that the matchings are identical when every school fills its capacity.

3. Conclusion

In this study, we have undertaken a comprehensive theoretical analysis of the Sequential Offer Mechanism (SOM), focusing intently on its defining properties of stability and strategy-proofness. Our inquiry reveals nuanced insights that both challenge and extend the current understanding of school choice mechanisms.

Our first key finding, as illustrated in Proposition 1, indicates that SOM does not guarantee a stable matching. This lack of guaranteed stability unveils a critical limitation in the mechanism, suggesting that participants, both schools and students, may have incentives to deviate from their initial assignments. This insight offers a cautionary note for policy-makers who may be considering SOM for educational placement, as unstable matchings can compromise the longevity and reliability of the assignment process.

Our second significant discovery, as set forth in Proposition 2, posits that under the condition of each school reaching its full capacity, SOM produces an outcome identical to that of the well-known Boston Mechanism. This convergence between SOM and the Boston Mechanism is intriguing, as it raises questions about the necessity or benefits of one mechanism over the other when the condition of full capacity is met. Moreover, this finding contributes to the ongoing discourse that compares different allocation mechanisms, emphasizing that the specifics of how admissions are conducted can indeed influence the nature of the matchings produced.

Our work serves as a foundational step in dissecting the complexities inherent in the SOM, thereby adding a new dimension to the expansive literature on market design and matching theory. While we have shed light on pivotal aspects of SOM, further research is needed to explore the behavioral implications of these theoretical findings. For instance, future studies might consider the impact of partial information, deadlines, or the introduction of waiting lists as factors that could influence the performance and attractiveness of SOM.

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