The Research on the Black-Scholes Formula

Peng Yun Wang，Guan Ying Dai，Wei Xin Zhang

（NanKai University TianJin 300071）

Abstract: At the beginning of 1970s，Fischer Black and Myron Scholes have made the unprecedented work in the domain of the option pricing theory，and proposed the first complete option pricing model，which named the Black—Scholes pricing formula . It has been accepted by the theoretical and industrial world for its wide application，and becomes the secondary great revolution in the financial domain.

However，in the actual financial market，a mass of finance practice has indicated that there is a serious warps between the hypothesis of Black-Scholes model and the actual markets.

This dissertation introduces the option elementary knowledge briefly，and then carries out a specific study on the the history of option pricing theory as well as factors that influence option price.

Keywords: Black-Scholes formula, Volatility smile, Traders' beliefs, the probability of positive returns

1. Black-Scholes Option Pricing Theory

ITO discovered a very important theorem in 1951, the ITO theorem (Ito's Lemma). Suppose variables x follow ITO process:

$$dS = \mu S dt + \sigma S dz \qquad (1.1)$$

Suppose f is a price function of financial derivative securities dependent on S, so from the ITO theorem:

$$\boldsymbol{df} = (\frac{\partial f}{\partial S}\boldsymbol{\mu S} + \frac{\partial f}{\partial t} + \mathbf{0.5}\frac{\partial^2 f}{\partial^2 S}\boldsymbol{\partial^2 S^2})\boldsymbol{dt} + \frac{\partial f}{\partial S}\sigma S dz \qquad (1.2)$$

Where dz is the same Wiener process, so G follows ITO process

Its drift rate is as follows:

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \mathbf{0.5}\frac{\boldsymbol{\partial^2 G}}{\boldsymbol{\partial x^2}}$$

Its variance rate is:

$$(\frac{\partial G}{\partial x})^2\boldsymbol{b^2}$$

Replace $\boldsymbol{S}$ with $\boldsymbol{\mu S}$, $\boldsymbol{\sigma^2}$ with $\boldsymbol{\sigma^2 S^2}$,we can get the results are:

$$\frac{dS}{S} = \boldsymbol{\mu dt + \sigma dz} \qquad (1.3)$$

$\boldsymbol{\sigma}$ is Stock Price Volatility; $\boldsymbol{\mu}$ is Stock Expected Return.

$\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are constants

From ITO theorem, we can know that the function G of $\boldsymbol{S}$ and t follows the following process:

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dz \qquad (1.4)$$

Now, we will deduce the differential equation of Black Scholes option pricing.

First, we assume that the stock price s satisfies the Wiener process as follows:

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}z \qquad (1.5)$$

Suppose $\boldsymbol{f}$ is a price function of financial derivative securities dependent on $\boldsymbol{S}$, so from the ITO theorem:

$$\boldsymbol{df} = (\frac{\partial f}{\partial S}\boldsymbol{\mu S} + \frac{\partial f}{\partial t} + \mathbf{0.5}\frac{\partial^2 f}{\partial^2 S}\boldsymbol{\partial^2 S^2})\boldsymbol{dt} + \frac{\partial f}{\partial S}\sigma S dz \qquad (1.6)$$

The discrete forms of equations (1.5) and (1.6) are:

$$\Delta S = \mu S \Delta t + \sigma S \Delta t \qquad (1.7)$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z \qquad (1.8)$$

Δf and ΔS are the changes of S and f after the same short time interval Δt respectively.

In order to eliminate the influence of Wiener process in this process, we must join in an appropriate portfolio consisting of certain stocks and financial derivatives.

The appropriate portfolio should be:

$-\mathbf{1}$: Financial derivative securities

$+\frac{\partial f}{\partial S}$:Stocks

The holder of this portfolio sells a financial derivative and then buys a quantity of $\frac{\partial f}{\partial S}$ stocks. We can define the value of a portfolio as Π . According to the definition, we can get:

$$\Pi = -\boldsymbol{f} + \frac{\partial f}{\partial S}\boldsymbol{S} \qquad (1.9)$$

The change in the value of the portfolio after the time is as follows:

$$\Delta\Pi = -\Delta\boldsymbol{f} + \frac{\partial f}{\partial S}\Delta\boldsymbol{S} \qquad (1.10)$$

By substituting equations (1.7) and (1.8) into equation (1.10), we get the following results:

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\boldsymbol{\sigma^2 S^2}\right)\Delta\boldsymbol{t} \qquad (1.11)$$

We assume that in the case of risk-free arbitrage, and the given risk-free interest rate is a constant, so it will be the same value for all maturities.

Therefore, the instantaneous return of this portfolio is similar to that of other short-term risk-free securities. The results are as follows:

$$\Delta\Pi = \boldsymbol{r} \cdot \Pi \cdot \Delta \mathrm{t} \qquad (1.12)$$

From equations (1.9) and (1.11), it can be obtained that:

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{t}} + \boldsymbol{rS}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{S}} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\sigma^2 S^2}\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{S}^2} = \boldsymbol{rf}$$

The above formula is the Black Scholes differential equation when we adopt financial derivative securities under various conditions.

The solution of the equation is not unique when it is defined by the underlying variable. However, when we define a specific financial derivative security, we can use boundary conditions to get a unique and definite value. It is because that these boundary conditions determine the value of financial derivatives in all possible boundary conditions.

The boundary conditions of European call options are as follows:

When $\boldsymbol{t} = \boldsymbol{T}$ $\quad \boldsymbol{f} = \boldsymbol{max}(\boldsymbol{S} - \boldsymbol{X}, \mathbf{0})$

The boundary conditions of European put options are as follows:

When $\boldsymbol{t} = \boldsymbol{T}$ $\quad \boldsymbol{f} = \boldsymbol{max}(\boldsymbol{X} - \boldsymbol{S}, \mathbf{0})$

$\boldsymbol{S}$ is the price of the stock at $\boldsymbol{T}$, and $\boldsymbol{X}$ is the strike price, that is, the execution price. Then we get the price of the European call option:

$$\mathrm{c} = \boldsymbol{SN}(\boldsymbol{d_1}) - \boldsymbol{Xe}^{-r(T-t)}\boldsymbol{N}(\boldsymbol{d_2})$$

$$\boldsymbol{d_2} = \boldsymbol{d_1} = \frac{\ln\left(\frac{\boldsymbol{S}}{\boldsymbol{X}}\right) + \left(\boldsymbol{r} - \frac{\boldsymbol{\sigma^2}}{\mathbf{2}}\right)(\boldsymbol{T} - \boldsymbol{t})}{\boldsymbol{\sigma}\sqrt{\boldsymbol{T} - \boldsymbol{t}}} = \boldsymbol{d_1} - \boldsymbol{\sigma}\sqrt{\boldsymbol{T}}$$

From the above formula, we can see that any variable that is easily affected by the risk preference of investors will be affected. This is a very important property of Black Scholes differential equation. From this equation, we can also see that the variables in the equation and risk preference are in a relatively independent state. The Black Scholes differential equation shows that the variables in the equation are independent of risk preference.

It plays a very important role in our future research and empirical research. If there is no risk preference in the Black Scholes differential equation, then the risk preference will not have any effect on its solution. Therefore, when we price $\boldsymbol{f}$, we can use any kind of risk preference at will.

In a risk neutral world, the expected value of the maturity date of European call options is as follows:

$$\boldsymbol{E}[\boldsymbol{max}(\boldsymbol{S_T} - \boldsymbol{X}, \mathbf{0})]$$

From the discussion of risk neutral pricing, we can see that the price $\boldsymbol{c}$ of European call options is the result of discounting this value at risk-free interest rate:

$$\boldsymbol{c} = \boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})}\boldsymbol{E}[\boldsymbol{max}(\boldsymbol{S_T} - \boldsymbol{X},\boldsymbol{0})]$$

2. Definition of Black Scholes option pricing bias

If the natural logarithm of a variable obeys a normal distribution, we call it a lognormal distribution. Using ITO theorem to deduce the process of $\boldsymbol{lnS}$

For $= \ln \boldsymbol{S}$, $\frac{\partial G}{\partial S} = \frac{1}{S}, \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial G}{\partial t} = \boldsymbol{0}$ then we can get the $\boldsymbol{G}$ progress:

$$\boldsymbol{dG} = \left(\boldsymbol{\mu} - \frac{\boldsymbol{\sigma}^2}{2}\right)\boldsymbol{dt} + \boldsymbol{\sigma dz}$$

Because $\boldsymbol{\mu}$ and are $\boldsymbol{\sigma}$ constants, we can know that G follows Wiener process.

Because of the constant drift rate $(\boldsymbol{\mu} - \boldsymbol{\sigma}^2)/2$ and constant variance rates $\boldsymbol{\sigma}^2$. Therefore, according to the relevant conclusions of the Wiener process, we can find that the G changes between the t and the future is normal distribution.[1]

Mean: $\left(\boldsymbol{\mu} - \frac{\sigma^2}{2}\right)(\boldsymbol{T} - \boldsymbol{t})$ Variance: $\boldsymbol{\sigma}^2(\boldsymbol{T} - \boldsymbol{t})$

Therefore, the change $\boldsymbol{G}$ during $\boldsymbol{T} - \boldsymbol{1}$ is: $\ln \boldsymbol{S_T} - \ln \boldsymbol{S}$

So, we can get:

$$\ln \boldsymbol{S_T} \sim \boldsymbol{\varphi}\left[\ln \boldsymbol{S} + \left(\boldsymbol{\mu} - \frac{\sigma^2}{2}\right)(\boldsymbol{T} - \boldsymbol{t}), \boldsymbol{\sigma}\sqrt{\boldsymbol{T} - \boldsymbol{t}}\right] (2.1)$$

From the above formula, we can know that the $\boldsymbol{S_T}$ satisfies the characteristics of logarithmic distribution. Moreover, the standard deviation of the $\ln \boldsymbol{S_T}$ and $\sqrt{\boldsymbol{T} - \boldsymbol{t}}$ are also proportional.

From equation (2.1) and some peculiar properties of the lognormal distribution we have known, we can get $\boldsymbol{E}(\boldsymbol{S_T})$ as follows:

$$\boldsymbol{E}(\boldsymbol{S_t}) = \boldsymbol{S}\boldsymbol{e}^{\boldsymbol{\mu}(\boldsymbol{T}-\boldsymbol{t})}$$

$\boldsymbol{\mu}$ is defined as the expected rate of return. So $\boldsymbol{var}(\boldsymbol{S_T})$ can be expressed as:

$$\text{var}\ (\boldsymbol{S_t}) = \boldsymbol{S}^2\boldsymbol{e}^{2\boldsymbol{\mu}(\boldsymbol{T}-\boldsymbol{t})}\left(\boldsymbol{e}^{\boldsymbol{\sigma}^2(\boldsymbol{T}-\boldsymbol{t})} - \boldsymbol{1}\right)$$

When we price European stock options, we generally assume that the process of stock price change is in accordance with the lognormal distribution. However, in the empirical process, the change process of stock price we encounter is not strictly satisfied with the lognormal distribution, and there is a certain deviation. Thus, it does not satisfy our ideal hypothesis and model. Below, we will discuss, analyze and study the effect of stock deviation from the perspective of lognormal distribution.

The true distribution may not be the same as the lognormal distribution. If we use Black-Scholes model to price options, there will be a pricing deviation. As we consider call options and put options in a large depth of virtual value, their value will be determined by the curves of different parts, and the B1ack-Scholes model will underestimate or overestimate their price, resulting in various deviations.[2]

We will use the parity relationship between call options and put options to obtain the deviation of option pricing in order to obtain real value:

$$\boldsymbol{p} + \boldsymbol{S} = \text{c} + \boldsymbol{X}\boldsymbol{e}^{-\boldsymbol{r}(\boldsymbol{T}-\boldsymbol{t})}$$

Assuming that the European call option with c price is now in a virtual state, the corresponding price of the European put option in the real state can be obtained, and its value is $\boldsymbol{p}$, and vice versa.

3. Design of Option Pricing Revision Model

3.1 Drawback of Black-Scholes models

Rubinstein,Hull and White,Longstaff state that the option is priced with the underlying asset with a jump, so that the value and the real value can not be equalized. Another source of bias exists in the real market: random dividends on stock returns, transaction taxes and transaction costs, and so on. These factors may also lead to a series of deviations between the price and the real value of the option obtained by Black-Scholes option pricing model. At the same time, traders will predict the expected value of volatility and the mean value of dividend as the deviation of volatility smile. Given the limitations of knowledge and foresight, heterogeneous beliefs and different learning mechanisms of traders, the different effects on option pricing are brought, which makes the Black-Scholes option pricing formula produce a series of deviations in empirical evidence.[3]

3.2 Determination of Option Pricing Revision Model

3.2.1 Probability of positive returns

The movement of the price of the underlying stock is in accordance with the geometric Brownian motion:

$$\frac{\mathrm{d}\boldsymbol{St}}{\boldsymbol{St}} = \boldsymbol{\mu}\mathrm{dt} + \boldsymbol{\sigma}\mathrm{dWt} \qquad (4.1)$$

$\boldsymbol{St}$ is the price of the stock at $\boldsymbol{t}$;
$\boldsymbol{\mu}$ is the stock return growth rate;
$\boldsymbol{\sigma^2}$ is the variance of stock returns;
$\{\boldsymbol{W_t},\boldsymbol{t} \in [\boldsymbol{0},\boldsymbol{T}]\}$is the standard vier process;
$\boldsymbol{T}$ is due
By solving (4.1), we can conclude that:

$$\boldsymbol{S_t} = \boldsymbol{S_0}\boldsymbol{e}^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t} \qquad (4.2)$$

So we can get a European call option in return:

$$(\boldsymbol{S_T} - \boldsymbol{K})^+ - \boldsymbol{C}\boldsymbol{e}^{rT} \qquad (4.3)$$

Among them: $\boldsymbol{r}$ is risk-free rate
So as the holder of European call option, the income should be:

$$(\boldsymbol{S_T} - \boldsymbol{K})^+ = \boldsymbol{max}\{\boldsymbol{0},\boldsymbol{S_T} - \boldsymbol{K}\} \qquad (4.4)$$

The probability of positive return of call option is:

$$\boldsymbol{p}\{(\boldsymbol{S_T} - \boldsymbol{K})^+ - \boldsymbol{C}\boldsymbol{e}^{rT} \geq \boldsymbol{0}\} = \boldsymbol{1} - \boldsymbol{N}(\boldsymbol{e_1}) \qquad (4.5)$$

$$\boldsymbol{e_1} = \frac{\ln\left(\frac{\boldsymbol{K} + \boldsymbol{C}\boldsymbol{e}^{rT}}{\boldsymbol{S_0}}\right) - \left(\boldsymbol{\mu} - \frac{\boldsymbol{\sigma^2}}{\boldsymbol{2}}\right)\boldsymbol{T}}{\boldsymbol{\sigma}\sqrt{\boldsymbol{T}}}$$

3.2.2 Investor beliefs

Investors can learn about the positive return probability of stocks through historical data. In other words, an investor can get the probability of a positive return from a particular market situation through related calculations. In order to make a certain profit, as the owner of the option, he would like to sell out an option when the probability of abstaining the positive return is far below his expectations;, and he would like to buy in an option when the probability of obtaining a positive return is far beyond his expections.

Let $\boldsymbol{p}$ be the positive return probability of the option holder so we can get:

$$\boldsymbol{D} = \boldsymbol{S_0 e}^{\left(\mu - r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N^{-1}(1-p)} - \boldsymbol{K e^{-rT}} \qquad (4.6)$$

Among them, the market price $\boldsymbol{C}$ of the holder of the option is :

$$\boldsymbol{C} = \begin{cases} \boldsymbol{D}, \text{ if } (\boldsymbol{S_0} - \boldsymbol{K e^{-rT}})^+ \leq \boldsymbol{D} \leq \boldsymbol{S_0} \\ (\boldsymbol{S_0} - \boldsymbol{K e^{-rT}})^+, \text{if } (\boldsymbol{S_0} - \boldsymbol{K e^{-rT}})^+ \geq \boldsymbol{D} \\ \boldsymbol{S_0}, \text{ if } \boldsymbol{D} \geq \boldsymbol{S_0} \end{cases}$$

3.2.3 Deviation of Black-Scholes formula

The holder of the option wants to buy an option and the option has a positive return probability of more than 0.5, and when the price of a European call option is below the equilibrium price, we can get:

$$\mathrm{C} < \boldsymbol{S_0 e}^{\left(\mu - r - \frac{\sigma^2}{2}\right)T} - \boldsymbol{K e^{-rT}} \qquad (4.7)$$

The holder of the option wants to buy an option and the option has a positive return probability of less than 0.5 , and when the price of a European call option is higher than the equilibrium price, we can get:

$$\mathrm{C} > \boldsymbol{S_0 e}^{\left(\mu - r - \frac{\sigma^2}{2}\right)T} - \boldsymbol{K e^{-rT}} \qquad (4.8)$$

The holder of the option wants to buy an option and the option has a positive return probability of just 0.5 , and when the price of a European call option is the equilibrium price, we can get:

$$\mathrm{C} = \boldsymbol{S_0 e}^{\left(\mu - r - \frac{\sigma^2}{2}\right)T} - \boldsymbol{K e^{-rT}} \qquad (4.9)$$

Black-Scholes option pricing model, the option price is expressed as:

$$\boldsymbol{C} = \boldsymbol{S_0 N(d_1)} - \boldsymbol{K e^{-rT} N(d_2)}$$

$$\boldsymbol{d_1} = \frac{\ln\left(\frac{\boldsymbol{S_0}}{\boldsymbol{K}}\right) + \left(\boldsymbol{r} + \frac{\boldsymbol{\sigma^2}}{\boldsymbol{2}}\right)\boldsymbol{T}}{\boldsymbol{\sigma\sqrt{T}}}$$

$$\boldsymbol{d_2} = \boldsymbol{d_1} - \boldsymbol{\sigma\sqrt{T}}$$

From this, we can confirm the deviation type of Black-Scholes option pricing formula from different empirical evidence. Given that stock prices follow the geometric brownian movement, traders' limitations in knowledge and foresight, heterogeneous beliefs, and learning mechanisms, the bias in Black-Scholes option pricing formulas can be observed from different market conditions. In the numerical study, the deviation type in this model can be consistent with the empirical level.

4. Simulation deviation analysis

4.1.1When the value of (μ, K, σ^2) in the B-S price is lower than the market price

$$\boldsymbol{S_0} = \boldsymbol{100}, \boldsymbol{r} = \boldsymbol{6\%}, \boldsymbol{T} = \boldsymbol{90}$$

4.1.2When the value of (μ, K, T) in the B-S price is lower than the market price

$$S_0 = 100, r = 6\%, \sigma^2 = 0.2^2$$

5. Summary and Outlook

Based on the limitations of knowledge and foresight and the lack of heterogeneous beliefs and learning mechanisms, there are obstacles to a reasonable pricing of options. As a result, Black-Scholes option pricing formula will also lead to deviation in practice. In concrete evidence, we can see that the deviation of Black-Scholes option pricing model is related to three factors:

(1) The deviation is related to the final price $\boldsymbol{K}$. The valuation of the in-price option model is higher than the market price of the option; Or the valuation of the in-price option model is lower than the market price of the option.

(2) Deviations are related to $\boldsymbol{T}$ .The valuation of the model near maturity is lower than the market price of the option.

(3) Deviations are related to volatility $\boldsymbol{\sigma}$. When the variance of the underlying asset is estimated to be high, the valuation of the model is often higher than the market price of the option.

In the modified model, we can see that the relevant deviations are properly avoided .

Of course, there are some defects in the article. For example, this paper does not consider the situation in which the underlying stock can pay dividends or the situation in which the call option can be exercised at any time before the specified expiration date. Therefore, the constant elasticity of variance diffusion process may provide a framework for ordinary stocks and other financial instruments to analyze and study the non-stationary returns.

References

[1] ZHOU Juan , HAN Li2yan,(2008) China Academic Joernal Electronic Publishing house, Recovering of implied risk2neutral probability distributions of foreign exchange futures options and market sentiment,197-205

[2] E. V. Shuryak, Yad. Fiz. 18, 1302 (1973) [Sov. J. Nucl. Phys.18, 667 (1974)].

[3] Scott, L.O., \Option Pricing When the Variance Changes Randomly: Theory, Estimation, and An Application," *Journal of Financial and Quantitative Analysis,* 22, 419-438, 1987.